223. Some Notes on the Cluster Sets of
Meromorphic Functions

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(Comm. by Kinjiro Kunugi, M.J.A., Nov. 12, 1966)

1. Let $D$ be a domain in the $z$-plane, $\Gamma$ its boundary, $E$ a totally disconnected compact set on $\Gamma$ and $z_0$ a point of $E$ such that $U(z_0) \cap (\Gamma - E) \neq \emptyset$ for any neighborhood $U(z_0)$ of $z_0$. We consider a normal exhaustion $\{F_n\}$ of the complementary domain $F$ of $E$ with respect to the extended $z$-plane and the graph $0 < u < R, 0 < v < 2\pi$ associated with this exhaustion in Noshiro's sense [3], where $R$ is the length of this graph and may be infinite. The niveau curve $u(z) = r(0 < r < R)$ on $F$ consists of a finite number of closed analytic curves $\gamma_i(r)(i = 1, 2, \cdots, m(r))$ and we set

$$A(r) = \max_{1 \leq i \leq m(r)} \int_{\gamma_i(r)} dv.$$ 

Now suppose that there exists an exhaustion $\{F_n\}$ with the graph satisfying

$$\lim_{r \to R} \frac{1}{A(r)} \int_0^r dr = \infty.$$ 

Then the integral $\int_0^R \exp \left( \frac{2\pi}{A(r)} \int_0^r dr \right) dr$ diverges, so that the complementary domain $F$ of $E$ belongs to the class $O_{ab}$ (see Kuroda [1]), i.e., $E$ belongs to the class $N^a_{ab}$ in the sense of Noshiro [4]. Therefore, for any single-valued meromorphic function $w = f(z)$ in $D$, the set $\Omega = C_p(f, z_0) - C_{\Gamma - E}(f, z_0)$ is empty or open and each value $\alpha$ belonging to $\Omega - R_\Omega(f, z_0)$ is an asymptotic value of $w = f(z)$ at $z_0$ or there is a sequence of points $\zeta_n \in E$ tending to $z_0$ such that $\alpha$ is an asymptotic value of $f(z)$ at each $\zeta_n$. Further $\Omega - R_\Omega(f, z_0)$ is an at most countable union of sets of the class $N^a_{ab}$. (These three facts have been proved by Noshiro in his recent paper [4].) We shall restrict our consideration to the case where $E$ is contained in a single boundary component $\Gamma_0$ of $\Gamma$. Then we have

**Theorem 1.** Suppose that $\Omega$ is not empty. If $E$ is contained in a single boundary component $\Gamma_0$ of $\Gamma$ and there exists an exhaustion $\{F_n\}$ with the graph satisfying (1), then $w = f(z)$ takes on every value, with two possible exceptions, belonging to any component $\Omega_n$ of $\Omega$, infinitely often in the intersection of any neighborhood of $z_0$ and $D$. 


In the special case where \( D \) is simply connected, we have

**Theorem 2.** Suppose that \( D \) is simply connected and \( w = f(z) \) is regular in the intersection of some neighborhood of \( z_0 \) and \( D \). Then, under the same assumptions as in Theorem 1, \( w = f(z) \) takes on every finite value, with one possible exception, belonging to any component \( \Omega_n \) of \( \Omega \) infinitely often in the intersection of any neighborhood of \( z_0 \) and \( D \).

**Remark 1.** If \( E \) is of logarithmic capacity zero, then there exists an exhaustion \( \{ F_n \} \) with the graph, its length being infinite. Hence the condition (1) is satisfied and we see that Theorems 1 and 2 are extensions of Noshiro's theorem [2].

2. We assume that \( E \) contains at least two points. Without any loss of generality, we may assume that an exceptional value \( w_o \) in \( \Omega_n \), if exists, is finite. Inside \( \Omega_n \) we draw a simple closed analytic curve \( C \) which does not pass through the point at infinity and encloses \( w_o \) and whose interior consists of only interior points of \( \Omega_n \). We select a positive number \( \gamma \) less than the diameter of \( \Gamma_0 \) such that \( f(z) \neq w_o \) in the common part of \( D \) and \( (K) \) and the closure \( M_0 \) of the union \( \bigcup C_p(f, \zeta) \) for all \( \zeta \) belonging to the intersection of \( \Gamma - E \) with \( (K) \) lies outside \( C \). We draw in \( (K) \) a simple closed analytic curve \( \gamma \) which encloses \( z_0 \) and does not pass through any point of \( E \). Since \( w_o \) is either an asymptotic value of \( w = f(z) \) at \( z_0 \) or there exists a sequence \( z_n \in E \) tending to \( z_0 \) such that \( w_n \) is an asymptotic value at each \( z_n \), it is possible to find a point \( z_n \) (may be \( z_0 \)) belonging to \( E \cap (\gamma) \), \( (\gamma) \) being the interior of \( \gamma \), such that \( w_n \) is an asymptotic value of \( w = f(z) \) at \( z_n \). Let \( A \) be the asymptotic path with the asymptotic value \( w_n \) at \( z_n \). We may assume that the image of \( A \) under \( w = f(z) \) is a curve lying completely inside \( C \). Considering the open set of points \( z \) in the intersection of \( D \) and \( (\gamma) \) such that \( w = f(z) \) lies inside \( C \), we denote by \( A \) its component containing the path \( A \). As is easily seen, the boundary of \( A \) consists of a finite number of arcs on \( \gamma \), at most a countable number of analytic curves (relative boundary) inside \( D \cap (\gamma) \), and a closed subset \( E_0 \) of \( E \).

Let \( r_0 \) be a fixed positive number such that for \( r_0 \leq r < R \) all the level curve \( \gamma_r \) does not intersect \( \gamma \) and does the asymptotic path \( A \). We take the component \( \gamma_\alpha \) of \( \gamma \) (one of \( \gamma_\beta(i = 1, 2, \ldots, m(r)) \)) enclosing \( z_0 \) and denote \( \Theta_\alpha \) the common part of \( \gamma_\alpha \) and the domain \( A; \Theta_\alpha \) consists of only a finite number of cross-cuts because we have taken \( \gamma \) less than the diameter of \( \Gamma_0 \). Denote by \( A(r) \) the common part of \( A \) and the exterior of \( \gamma_\alpha \), by \( A(r) \) the area of the Riemannian image of the open set \( A(r) \) under the function \( w = f(z) \) and by \( L^r(\alpha) \) the total length of the image of \( \Theta_\alpha \). Then, using the local parameter
\[ \zeta = u + iv, \text{ we have} \]
\[ L'(r) = \int_{\sigma_r} \frac{f'}{|f'|} \, dv. \]
Denote by \( \delta > 0 \) the distance of \( C \) from the image of \( A \). Then, a geometric consideration gives \( L'(r) \geq 2\delta \) for \( r_0 \leq r < R \) and by Schwarz's inequality, we have
\[ 4\delta^2 \leq L'(r)^2 \leq A(r) \int_{\sigma_r} \frac{f''}{|f'|} \, dv \leq A(r) \int_{u=r} \frac{f'}{|f'|} \, dv. \]
Note that
\[ \int_{u=r} \frac{f'}{|f'|} \, dv = \frac{dA(r)}{dr}. \]
From (2) and (3) we have
\[ 4\delta^2 \int_{r_0}^{r} \frac{dr}{A(r)} \leq A(r) - A(0), \]
so that our condition (1) gives
\[ \lim_{r \to R} A(r) = \infty. \]
Next we shall prove that the regularly exhaustibility condition in Ahlfors' sense is satisfied. Denoting by \( L(r) \) the total length of the image of \( \theta_r \), the common part of \( \gamma_r \) and \( \gamma \), we have
\[ L(r)^2 \leq 2\pi \frac{dA(r)}{dr}. \]
Now, contrary, suppose that
\[ \lim \inf_{r \to 0} \frac{L(r)}{A(r)} \geq \sigma > 0. \]
Then from (6) and (7) we see
\[ \frac{\sigma^2}{2\pi} (R-r) = \frac{\sigma^2}{2\pi} \int_{r}^{R} \frac{dr}{A(r)^2} \leq \int_{r}^{R} \frac{dA(r)}{A(r)^2} = \frac{1}{A(R)}, \]
since \( A(R) = \infty \) by (5). Using (4), we have thus
\[ 2\sigma^2 \delta^2 \frac{1}{(R-r)} \int_{r}^{R} \frac{dr}{A(r)} \leq 1. \]
This contradicts our condition (1) and the regularly exhaustibility condition must hold.

3. Now it is easy to prove our theorem. Indeed, we need only to follow Noshiro's arguments [2]. For completeness, we shall give proofs in the below. Because of Noshiro's theorem, it is enough for us to prove the theorem under the condition that \( E \) contains at least two points.

Proof of Theorem 1. Contrary to our assertion, we suppose that there are three exceptional values \( w_0, w_1, \) and \( w_2 \) in \( \Omega_n \), where it does not bring any loss of generality if we assume these three values are finite. Inside \( \Omega_n \) we draw a simple closed analytic curve \( C \) which encloses \( w_0, w_1, \) and passes through \( w_2 \) but not through the point at infinity and whose interior consists of only interior points.
of \( \Omega_a \). We select a positive number \( \eta \) less than the diameter of \( \Gamma_0 \) such that \( f(z) \neq w_0, w_1, \) and \( w_2 \) in the common part of \( D \) and \( \text{(K)}: |z-z_0|<\eta \) and the closure \( M_0 \) lies outside \( C \). We determine \( \gamma, A, \) and \( \delta \) by the same way as in \$2 \) and for them we take \( r_0 \). We shall show that \( \Delta \) is simply-connected. Note that the boundary of \( \Delta \) does not contain any closed analytic curve, since any analytic curve in the boundary of \( \Delta \) is transformed by \( w=f(z) \) into a curve lying on the simple closed curve \( C \) passing through the exceptional value \( w_2 \). Further, the boundary of the bounded domain \( \Delta \) consists of a single continuum, since \( E \) is contained in a single component \( \Gamma_0 \) of \( \Gamma \). Thus it is concluded that \( \Delta \) is simply connected. Now it is clear that the open set \( \Delta(r), r_0 \leq r < R \), consists of simply connected components, because \( \theta \) does not contain any loop-cut. We denote these components by \( \Delta^{(i)}(r) \) \( (i=1, 2, \ldots, p(r)) \). Denote by \( \Phi^{(i)}(r) \) the Riemannian image of \( \Delta^{(i)}(r) \) under \( w=f(z) \) \( (i=1, 2, \ldots, p(r)) \). If we denote by \( \theta_0 \) the domain obtained by excluding the two points \( w_0 \) and \( w_1 \) from the interior of \( C \), then, by hypothesis, \( \Phi^{(i)}(r) \) is a finite covering surface of the base surface \( \Phi_0 \) \( (i=1, 2, \ldots, p(r)) \). By Ahlfors’ principal theorem on covering surfaces, we have

\[
S^{(i)}(r) \leq hL^{(i)}(r) \quad (i=1, 2, \ldots, p(r)),
\]

where \( S^{(i)}(r) \) denotes the average number of sheets of \( \Phi^{(i)}(r) \), i.e., \( S^{(i)}(r) \) denotes the ratio between the area of \( \Phi^{(i)}(r) \) and the area of \( \Phi_0 \), \( L^{(i)}(r) \) the length of the boundary of \( \Phi^{(i)}(r) \) relative to \( \Phi_0 \), and \( h \) is a constant dependent only upon \( \Phi_0 \). From (10)

\[
\sum_{i=1}^{p(r)} S^{(i)}(r) \leq h \sum_{i=1}^{p(r)} L^{(i)}(r),
\]

that is,

\[
S(r) \leq h(L(r)+L_0),
\]

where \( L_0 \) denotes the total length of the image of the arcs of \( \gamma \) included in the boundary of \( \Delta \). Accordingly

\[
\liminf_{r \to R} \frac{L(r)}{S(r)} \geq \frac{1}{h} > 0,
\]

while we have showed in \$2 \) that the regularly exhaustibility condition holds. Contradiction. Our theorem must be true.

**Proof of Theorem 2.** Suppose that there are two finite exceptional values \( w_0 \) and \( w_1 \) within \( \Omega_a \), and let \( C \) be any simple closed analytic curve in \( \Omega_a \), which surrounds \( w_0 \) and \( w_1 \) and whose interior consists of only interior points of \( \Omega_a \). Let \( \Delta \) be the domain defined in the same way as in the proof of Theorem 1. Then, we can easily see that \( \Delta \) is also simply connected, for if \( \Delta \) were not simply connected, the boundary of \( \Delta \) would contain at least one closed analytic contour \( q \) such that \( q \) is a loop-cut of \( D \). Hence \( w=f(z) \) would take inside \( q \) a value lying outside the simple closed curve.
while \( w = f(z) \) is regular both inside and on \( q \), and the image of \( q \) by \( w = f(z) \) would lie on \( C \). This is a contradiction. Repeating the same argument as in the proof of Theorem 1, we complete the proof.

4. In this section we shall give an example of \( E \) satisfying the condition (1) by means of Cantor sets. We prove.

**Theorem 3.** Let \( E \) be a Cantor set on the interval \( I_o: [-1/2, 1/2] \) on the real axis of the \( z \)-plane with successive ratios \( \xi_n, 0 < \xi_n = 2l_n < 2/3 \). If
\[
\limsup_{n \to \infty} \left( \sum_{p=0}^{\infty} \frac{\log \xi_p^{-1}}{2^p} \right) \left( \sum_{p=1}^{\infty} \frac{\log \xi_p^{-1}}{2^p} \right) = \infty,
\]
then there exists an exhaustion \( \{F_n\} \) of the complementary domain \( F \) of \( E \), the graph associated with which satisfies the condition (1).

**Proof.** Defining the Cantor set \( E \), we repeat successively to exclude an open segment from the center of another segment and there remain \( 2^n \) segments of equal length \( [l^n_i, l^n_{i+1} \rangle \) after we repeat \( n \) times, beginning with the interval \( I_o \). We denote by \( I_{n,k} (n=1, 2, \ldots; k=1, 2, \ldots, 2^n) \) these segments and by \( C_{n,k} \) \( (n=1, 2, \ldots; k=1, 2, \ldots, 2^n) \), the circles \( |z-z_{n,k}| = (\prod_{n=0}^{k-1} l_n) (1-l_n)/2 \), where \( z_{n,k} \) are the middle points of \( I_{n,k} \). Supposing that \( C_{n,k} \) encloses \( C_{n+1,k-1} \) and \( C_{n+1,k} \), we see that these two circles touch outside each other, and denote by \( S_{n,k} \) \( (n=1, 2, \ldots; k=1, 2, \ldots, 2^n) \) the ring domains bounded by \( C_{n,k} \) and \( C_{n+1,k-1} \cup C_{n+1,k} \). The harmonic modulus \( \mu_n \) of \( S_{n,k} \) is greater than \( \log (2\xi_n^{-1}/3) \). We define an exhaustion \( \{F_n\} \) of \( F \) as follows. The outside of the circle \( |z|=2 \) is taken as \( F_0 \) and the common part of the outsides of all the \( C_{n,k} \) \( (k=1, 2, \ldots, 2^n) \) is taken as \( F_n \). Then, for each \( n \), the open set \( F_{n+1}-F_n \) consists of ring domains \( S_{n,k} \) \( (k=1, 2, \ldots, 2^n) \), so that its harmonic modulus \( \sigma_n \) is equal to \( \mu_n/2^n \). Hence the length \( R \) of the graph associated with this \( \{F_n\} \) is
\[
\sum_{p=0}^{\infty} \sigma_p = \sum_{p=0}^{\infty} \frac{\mu_p}{2^p},
\]
where \( \sigma_0 = \mu_0 \) is the harmonic modulus of the ring domain \( F_1-F_0 \).

It is easily seen that
\[
A(r) = \frac{2\pi}{2^n} \quad \text{if} \quad \sum_{p=0}^{n} \sigma_p < r \leq \sum_{p=0}^{n+1} \sigma_p.
\]
Hence, if \( r = \sum_{p=0}^{n} \sigma_p \), then
\[
(14) \quad R-r = \sum_{p=n+1}^{\infty} \frac{\mu_p}{2^p} \geq \sum_{p=n+1}^{\infty} \frac{\log \xi_p^{-1}}{2^p} + \frac{\log (2/3)}{2^n}
\]
and
\[
(15) \quad \int_0^r \frac{dr}{A(r)} = \frac{1}{2\pi} \sum_{p=0}^{n} \frac{\sigma_p}{2^n} < \frac{1}{2\pi} \sum_{p=0}^{n} \frac{\mu_p}{2^n} + \frac{1}{2\pi} \sum_{p=1}^{n} \log \xi_p^{-1} + \frac{n}{2\pi} \log (2/3).
\]
Therefore it is enough for us to show that the condition (1) holds.
when \( R < \infty \). Then \((\log (2/3))^{n/2^{n+1}\pi} \to 0\) as \( n \to \infty \) and
\[
\left| \left(\log (2/3)\right)(1/2^{n+1}\pi)\sum_{p=1}^{n} \log \frac{1}{\xi_{p}^{-1}} \right| \leq \log (2/3) | R/2\pi = O(1) \]
Further we see from \((13)\) that \( n/\sum_{p=1}^{n} \log \frac{1}{\xi_{p}^{-1}} \to 0\) as \( n \to \infty \). Hence,
from \((14)\) and \((15)\),
\[
(R-r) \int_{0}^{r} \frac{dx}{A(r)} \geq \left( \sum_{p=k+1}^{n} \frac{\log \frac{1}{\xi_{p}^{-1}}}{2^{p}} \right) \left( \sum_{p=1}^{n} \log \frac{1}{\xi_{p}^{-1}} \right) \left( \frac{1}{2\pi} (1-o(1)) \right) + O(1),
\]
so that, by making \( n \to \infty \), we see that the condition \((1)\) holds.

**Example.** If successive ratios \( \xi_{n} \) satisfy
\[
(16) \quad \xi_{n+1} = O(\xi_{n}^{\lambda}) \quad \text{with} \quad \lambda > \sqrt{2} \quad \text{and} \quad n = 1, 2, \ldots,
\]
then they satisfy \((13)\).

**Remark 2.** It is well-known that a Cantor set \( E \) is of logarithmic capacity zero if and only if
\[
\sum_{p=1}^{n} \frac{\log \frac{1}{\xi_{p}^{-1}}}{2^{p}} = \infty.
\]
Hence we see that there exist ones of positive logarithmic capacity among Cantor sets satisfying \((16)\) for \( \lambda, 2 > \lambda > \sqrt{2} \).

**References**