128. A Milnor Conjecture on Spin Structures

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Let \( \xi \) denote a principal \( SO(n) \)-bundle over a \( CW \)-complex \( B \) and let \( E(\xi) \) denote the total space of \( \xi \). A spin structure on \( \xi \) is a pair \((\eta, f)\) which satisfies

1. A principal bundle \( \eta \) over \( B \) with the spinor group \( \text{Spin}(n) \) as structural group; and
2. A map \( f : E(\eta) \to E(\xi) \) such that the following diagram is commutative.

\[
\begin{align*}
E(\eta) \times \text{Spin}(n) \ar[r]^{f \times 1} & E(\eta) \ar[d]^f \ar[r] & B. \\
E(\xi) \times \text{SO}(n) \ar[r] & E(\xi)
\end{align*}
\]

Here \( \lambda \) denotes the standard homomorphism from \( \text{Spin}(n) \) to \( \text{SO}(n) \) and horizontal lines denote the right translation. A second spin structure \((\eta', f')\) on \( \xi \) is identified with \((\eta, f)\) if there exists an isomorphism \( g \) from \( \eta' \) to \( \eta \) so that \( f \circ g = f' \). Then J. Milnor stated the following conjecture [1, pp. 198-203]:

If \((\eta, f)\) and \((\eta', f')\) are two spin structures on the same \( SO(n) \)-bundle, with \( n > \text{dim } B \), then \( \eta \) is necessarily isomorphic to \( \eta' \).

In this note we shall present the affirmative answer when \( B \) is compact connected. By Milnor we have the following

Lemma [1, p. 199]: If \( \xi \) admits a spin structure then the number of distinct spin structures on \( \xi \) is equal to the number of elements in \( H^1(B; \mathbb{Z}_2) \).

Now the following lemma is clear.

**Lemma 1.** If \( \xi \) admits two spin structures \((\eta, f)\) and \((\eta', f')\) such that \( \eta \) is isomorphic to \( \eta' \) then there exists a spin structure \((\eta, f'')\) on \( \xi \) which is isomorphic to \((\eta', f')\).

Let \( p_1 \) denote the projection map of the bundle \( \xi \). If two spin structures \((\eta_1, f_1), (\eta_2, f_2)\) are given, from \( p_1 = p_1 f_1 = p_1 f_2 \), we have a map \( g : E(\eta) \to \text{SO}(n) \) defined by \( f_1(x) = f_2(x) \cdot g(x) \) for \( x \in E(\eta) \). Here \( \cdot \) denotes the right translation. Clearly \( g \) satisfies \( g(x \cdot h) = \lambda(h)^{-1} \times g(x) \times \lambda(h) \) for \( h \in \text{Spin}(n) \) where \( \times \) denotes the group multiplication. Conversely \( g \) is a map as above and let \((\eta, f)\) be a spin structure on \( \xi \). Then \((\eta, f \cdot g)\) is also a spin structure on \( \xi \). And moreover let \( g' \) be another map such as \( g \). Then \((\eta, f \cdot g)\) is isomorphic to \((\eta, f \cdot g')\) if

1) Of course the map \( f \cdot g \) is defined by \( (f \cdot g)(x) = f(x) \cdot g(x) \).
and only if there exists a map \( \varphi : E(\eta) \to \text{Spin}(n) \) which satisfies \( \varphi(x \cdot h) = h^{-1} \times \varphi(x) \times h \) and \( g(x) = g'(x) \times \lambda(\varphi(x)) \). Now we define two groups \( \langle E(\eta), \text{SO}(n) \rangle \) and \( \langle E(\eta), \text{Spin}(n) \rangle \) as follows:

\[
\langle E(\eta), \text{SO}(n) \rangle = \{ g : E(\eta) \to \text{SO}(n), g(x \cdot h) = \lambda(h)^{-1} \times g(x) \times \lambda(h) \}
\]

\[
\langle E(\eta), \text{Spin}(n) \rangle = \{ \varphi : E(\eta) \to \text{Spin}(n), \varphi(x \cdot h) = h^{-1} \times \varphi(x) \times h \}.
\]

Obviously \( \lambda \) induces a homomorphism \( \lambda_* : \langle E(\eta), \text{Spin}(n) \rangle \to \langle E(\eta), \text{SO}(n) \rangle \) and if \( B \) is connected \( \lambda_* \) is injective. Let \( \langle \eta \rangle \) denote the set of spin structures on \( \xi \) having \( \eta \) as the bundle of structures. By the above argument we have

**Lemma 2.** The number of \( \langle \eta \rangle \) is equal to the number of cosets of \( \langle E(\eta), \text{SO}(n) \rangle \) by \( \lambda_* \langle E(\eta), \text{Spin}(n) \rangle \).

Let \( (\eta, f_0) \) be a spin structure on \( \xi \) and define the group

\[
\langle E(\xi), \text{SO}(n) \rangle = \{ \psi : E(\xi) \to \text{SO}(n), \psi(x \cdot g) = g^{-1} \times \psi(x) \times g \}.
\]

It is obvious that \( f_0 \) induces the homomorphism \( f_{0*} : \langle E(\xi), \text{SO}(n) \rangle \to \langle E(\eta), \text{SO}(n) \rangle \) defined by \( f_{0*}(\psi) = \psi \circ f_0 \). Since the kernel of \( \lambda \) is contained in the center of \( \text{Spin}(n) \) we have

**Lemma 3.** When \( B \) is compact \( f_{0*} \) is the isomorphism.

Now consider the inverse image of \( \lambda_* \langle E(\eta), \text{SO}(n) \rangle \) by \( f_{0*} \). Let \( \langle E(\xi), \text{SO}(n) \rangle \) denote the subgroup of \( \langle E(\xi), \text{SO}(n) \rangle \) consisting on elements which have a lifting: \( E(\xi) \to \text{Spin}(n) \). Then analogously to Lemma 3 we have

**Lemma 4.** \( f_{0*}\langle E(\xi), \text{SO}(n) \rangle = \lambda_* \langle E(\eta), \text{Spin}(n) \rangle \).

Combining Milnor's lemma with the above lemmas we have

**Lemma 5.** When \( B \) is compact and connected the number of elements of \( \pi_1(B, \mathbb{Z}) \) is equal to the product of the number of cosets of \( \langle E(\xi), \text{SO}(n) \rangle \) by \( \langle E(\xi), \text{SO}(n) \rangle \) with the number of bundles which give a spin structure on \( \xi \).

Let \( B_G \) denote the classifying space for a topological group \( G \) and let \( x_\xi \) denote the characteristic map \( B \to B_G \) for a \( G \)-bundle \( \xi \). The homomorphism \( \lambda : \text{Spin}(n) \to \text{SO}(n) \) usually induces the correspondence \( B_1 : \pi(B, B_{\text{Spin}(n)}) \to \pi(B, B_{\text{SO}(n)}) \). Then it is clear that the number of the inverse image of \( x_\xi \) by \( B_1 \) is equal to the number of bundles which give a spin structure on \( \xi \). If \( n \) is larger than \( \text{dim } B \), then \( \pi(B, B_{\text{SO}(n)}) \) and \( \pi(B, B_{\text{Spin}(n)}) \) are equal to \( \pi(B, B_{\text{SO}(n)}) \) and \( \pi(B, B_{\text{Spin}(n)}) \) respectively. Hence we give a group structure to \( \pi(B, B_{\text{Spin}(n)}) \) and \( \pi(B, B_{\text{SO}(n)}) \) so that \( B_1 \) is a homomorphism. These considerations show that the number of bundles which give a spin structure on \( \xi \) is independent on \( \xi \), therefore the number of cosets of \( \langle E(\xi), \text{SO}(n) \rangle \) by \( \langle E(\xi), \text{SO}(n) \rangle \) is also free from \( \xi \). That is to say the case is only necessary for our purpose that \( \xi \) is trivial. Now we suppose that \( \xi \) is trivial. Let \( \{ B, \text{SO}(n) \} \) denote the group consisting on all maps: \( B \to \text{SO}(n) \) and let \( \rho \) denote the standard cross-section: \( B \to E(\xi) \). It is
easily shown that the homomorphism \( \rho_* : \langle E(\xi), SO(n) \rangle \to \{ B, SO(n) \} \) is bijective where \( \rho_* \) is defined by \( \rho_*(\phi) = \phi \circ \rho \). Clearly \( \rho_* \langle E(\xi), SO(n) \rangle \) is contained in \( \lambda_*\{ B, Spin(n) \} \).

Conversely, for a map \( \lambda \varphi, \varphi : B \to Spin(n) \), define a map \( \phi : E(\xi) \to SO(n) \) by \( \phi(b, g) = g^{-1} \times \lambda(\varphi(b)) \times g \). Then \( \phi \) is an element of \( \langle E(\xi), SO(n) \rangle \) such that \( \rho_*(\phi) = \lambda \varphi \). Let \( \tilde{\varphi} \) be a map \( E(\xi) \to Spin(n) \) defined by \( \tilde{\varphi}(b, g) = h^{-1} \times \varphi(b) \times h \) for \( \lambda(h) = g \). Since the kernel of \( \lambda \) is contained in the center of \( Spin(n) \), \( \tilde{\varphi} \) is well defined and continuous. By \( \lambda \tilde{\varphi} = \phi \) we can know that \( \phi \) is an element of \( \langle E(\xi), SO(n) \rangle \), i.e., we have

**Lemma 6.** \( \rho_* \) is bijective and maps the subgroup \( \langle E(\xi), SO(n) \rangle \) onto the subgroup \( \lambda_*\{ B, Spin(n) \} \).

Let \( X_\iota^p \) denote the cohomology class of \( \mathcal{H}^q(SO(n) ; \mathbb{Z}_2) \) which represents the \( \mathbb{Z}_2 \)-bundle \( Spin(n) \to SO(n) \). Consider a homomorphism \( \phi : \{ B, SO(n) \} \to \mathcal{H}^q(B, \mathbb{Z}_2) \) defined by \( \phi(\phi) = \phi^*(X) \). Now we suppose that \( \Phi(\phi) = 0 \). It is known that if we identify \( \mathcal{H}^q(SO(n), \mathbb{Z}_2) \) with \( \text{Hom}(\pi_q(SO(n)), \pi_q(SO(n))) X \), is correspond to the identity. Since \( B \) is connected, we can also identify \( \mathcal{H}^q(B, \mathbb{Z}_2) \) with \( \text{Hom}(H_1(B), \pi_q(SO(n))) \). Then \( \Phi^*(X) \) is interpreted as the composite homomorphism:

\[
\mathcal{H}^q(B) \xrightarrow{\phi^*} \mathcal{H}^q(SO(n)) \xrightarrow{\pi_q(SO(n))} \pi_q(SO(n)).
\]

Hence \( \Phi(\phi) = 0 \) implies that the homomorphism \( \phi_* : \pi_q(B) \to \pi_q(SO(n)) \) is trivial, i.e., \( \phi \) can be lifted. Hence we have

**Lemma 7.** \( \Phi \) induces the injection:

\[
\{ B, SO(n) \} / \lambda_*\{ B, Spin(n) \} \to \mathcal{H}^q(B ; \mathbb{Z}_2).
\]

If \( n > \dim B \) we can take the real projective space \( PR^{n-1} \) as the classifying space for \( \mathbb{Z}_2 \)-bundles over \( B \). On the other hand [2, p. 97] there exists an imbedding \( P_n : PR^{n-1} \to SO(n) \) such that \( P_n^* : \mathcal{H}^q(SO(n) ; \mathbb{Z}_2) \to \mathcal{H}^q(PR^{n-1} ; \mathbb{Z}_2) \) is bijective. Thus we have

**Lemma 8.** \( \Phi : \{ B, SO(n) \} / \lambda_*\{ B, Spin(n) \} \to \mathcal{H}^q(B ; \mathbb{Z}_2) \) is bijective. From lemmas we obtain our main theorem.

**Theorem.** Let \( B \) be a compact connected CW-complex. If a principal \( SO(n) \)-bundle over \( B \) admits two spin structures \(( \eta, f ) \) and \(( \eta', f' ) \), with \( n > \dim B \), \( \eta \) is necessary isomorphic to \( \eta' \).

**References**


2) Non-zero element of \( \mathcal{H}^q(SO(n); \mathbb{Z}_2) \equiv \mathbb{Z}_2 \).