§ 1. Introduction. Let \( \Omega \) be a compact oriented Riemannian \( n \)-space with smooth boundary \( \Gamma \). Let \( A \) be a linear partial differential operator on \( \Omega \) of order \( 2m \). We assume \( A \) is strongly elliptic, that is, there is a constant \( C > 0 \) such that, for any \( x \) in \( \Omega \) and for any non zero vector \( \xi \) cotangent to \( \Omega \) at \( x \), we have
\[
C^{-1} |\xi|^{2m} \leq \text{Re} \sigma_{2m}(A)(x, \xi) \leq C |\xi|^{2m},
\]
where \( \sigma_{2m}(A) \) is the principal symbol of \( A \). We consider normal systems \( \{B_r\}_{r \in R} \), \( R = (r_1, r_2, \ldots, r_m) \), of \( m \) boundary operators \( B_{r_j} \). \( r_j \) is the order of \( B_{r_j} \). We assume \( r_j < 2m \) for any \( j = 0, 1, \ldots, m \). The problem to be considered is

**Problem 1.** Characterize those couples \( \{A, \{B_r\}_{r \in R}\} \) which give, with some constants \( 1/2 > \varepsilon \geq 0, C, \beta > 0 \), the estimate
\[
(1) \quad \text{Re}((A + \beta)u, u)_{L^2(\Omega)} \geq C \|u\|_{H^{m-1/2}(\Omega)}^2
\]
for all \( u \) in \( H^m_\varepsilon(\Omega) = \{u \in H^m(\Omega) ; B_r u \mid_{r=0} = 0, \text{for any } r \in R\} \).

Here \( H^s(\Omega) \) denotes the Sobolev space on \( \Omega \) of order \( s \), \( \| \cdot \|_{H^s(\Omega)} \) is its norm and \(( \cdot, \cdot )_{L^2(\Omega)} \) is the inner product in \( L^2(\Omega) \).

If \( 1/2 > \varepsilon \geq 0 \), the problem was treated in far stronger form in [3]. In this note we concern with the case \( \varepsilon = 1/2 \). So the problem is

**Problem 1'.** Characterize those couples \( \{A, \{B_r\}_{r \in R}\} \) which give, with some constants \( C, \beta > 0 \), the estimate
\[
(2) \quad \text{Re}((A + \beta)u, u)_{L^2(\Omega)} \geq C \|u\|_{H^{m-1/2}(\Omega)}^2
\]
for all \( u \) in \( H^m(\Omega) \).

We assume the following hypothesis (H) that was proved in the case \( 0 \leq \varepsilon < 1/2 \) necessary for the estimate (1) to hold. (See [3] and [6].)

(H) The set \( R \) coincides with one of the \( R_j \)'s defined by \( R_j = (0, 1, \ldots, m-j-1, m, m+1, \ldots, m+j-1) \), \( 1 \leq j \leq m \). Under this hypothesis we give a necessary and sufficient condition for the estimate (2) to hold.

Proofs are omitted. Detailed discussions will be published elsewhere.*

§ 2. Results. We denote by \( \nu \) the interior unit normal to \( \Gamma \) and

* This work was done during the author's stay in Paris. He expresses his hearty thanks to Professor J. L. Lions for his constant encouragement.
by $D_n$ the normal derivative $-i \frac{\partial}{\partial \nu}$ multiplied by $-i = -\sqrt{-1}$. $S_j$ is the complement of $R_j$ in the set \{0, 1, 2, \ldots, 2m-1\}. Then $B_r$, $r \in R_j$ can be written as

$$B_r = D_n - \sum_{p \leq r} B_{r-p} D_n,$$

where $B_{r-p}$ is a pseudo-differential operator on $\Gamma$ of order $\leq r - p$. Let $A = (1 - \Delta')^{1/2}$ where $\Delta'$ is the Laplace-Beltrami operator associated with the metric on $\Gamma$. Then $A^k$ is an isomorphism from $H^k(\Gamma)$ to $H^{k-\kappa}(\Gamma)$. $A^*$ denotes the formal adjoint of $A$.

We choose and fix $\alpha$ so large that we can solve uniquely the problem:

$$(A + A^* + 2\alpha) v = 0$$

and obtain the estimates, for any $s \in \mathbb{R}$,

$$C^{-1} \sum_{k=rrc-j}^{m-1} \| \phi_k \|^2_{H^{s-1/2} (\Gamma)} \leq \| v \|^2_{H^s(\Omega)} \leq C \sum_{k=rrc-j}^{m-1} \| \phi_k \|^2_{H^{s-1/2} (\Gamma)}.$$

Here and hereafter we denote by $C$ different constants $>0$ in different occurrences.

Now we fix $B = \{B_r\}_{r \in R_j}$. We decompose any $u$ in $H^m_\mathcal{B}(\Omega)$ into sum of two functions $v$ and $w$:

$$u = v + w,$$

where

$$(A + A^* + 2\beta) v = 0 \text{ on } \Omega, \quad D_n^k v|_r = D_n^k u|_r, \quad 0 \leq k \leq m - 1,$$

and $D_n^j w|_r = 0$, $0 \leq k \leq m - 1$. This implies that $D_n^k v|_r = 0$ for $0 \leq k \leq m - j - 1$. We set $D_n^k u|_r = A^k \phi_k$, $m - j \leq k \leq m - 1$. Let $H^m_\mathcal{B}(\Omega)$ be the closure of $H^m_\mathcal{B}(\Omega)$ in $H^m(\Omega)$. Then $H^m_\mathcal{B}(\Omega) = \{ u \in H^m(\Omega) : D_n^k u|_r = 0, 0 \leq k \leq m - j - 1 \}$. The decomposition (6) is a topological decomposition of $H^m(\Omega)$. (See [5].) Now we take any $u$ in $H^m_\mathcal{B}(\Omega)$. Then using the boundary condition $B_r u|_r = 0$ and the decomposition (6), we can find pseudo-differential operators $H_{p,q}$ on $\Gamma$ of order $2m - 1$, $m - j \leq p$, $q \leq m - 1$, such that

$$\text{Re}((A + A^* + 2\beta) u, w)_{L^2(\Gamma)} = \text{Re}((A + A^* + 2\beta) w, w)_{L^2(\Gamma)} + \sum_{p,q=rrc-j}^{m-1} (H_{p,q}(\beta) \varphi_q, \varphi_p)_{L^2(\Gamma)}.$$

(See [2].)

Let $T$ be the 1 dimensional circle $= \mathbb{R} / 2\pi \mathbb{Z}$. We consider the elliptic operator $\tilde{A} = A + D_n^m$, $s \in T$, on $\Omega \times T$ and boundary operators $\{B_r\}_{r \in R_j}$ on $\Gamma \times T$. $H^m_\mathcal{B}(\Omega \times T)$ denotes the closure in $H^m(\Omega \times T)$ of $H^m_\mathcal{B}(\Omega \times T) = \{ f \in H^m(\Omega \times T) : B_r f|_{\Gamma \times T} = 0, r \in R_j \}$. Decomposition corresponding to (6) holds for functions in $H^m_\mathcal{B}(\Omega \times T)$, that is, for any $f$ in $H^m_\mathcal{B}(\Omega \times T)$,
\[ f = g + h, \quad (\tilde{A} + \tilde{A}^* + 2\hat{\beta})g = 0 \text{ on } \Omega \times T, \]
\[ D_m^k f |_{\Gamma \times T} = D_m^k f |_{\Gamma \times T}, \quad 0 \leq k \leq m - 1.\]

We set \( D_m^k f |_{\Gamma \times T} = (\tilde{A})^k \phi_k, \) \( m - j \leq k \leq m - 1, \) where \( \tilde{A} = (1 - A^* + D^2)^{1/2}. \) Just as we did above, we can find pseudo-differential operators \( \tilde{H}_{pq}(\beta) \) on \( \Gamma \times T \) of order \( 2m - 1 \) such that for any \( f \) in \( H^m_{\beta}(\Omega \times T) \)

\[
\text{Re}((\tilde{A} + \beta)f, f)_{L^2(\Omega \times T)} = \text{Re}((\tilde{A} + \beta)h, h)_{L^2(\Omega \times T)} + \sum_{p,q=m-j}^{m-1} (\tilde{H}_{pq}(\beta) \phi_q, \phi_p)_{L^2(\Gamma \times T)}.
\]

Our first result is

**Theorem 1.** Each of the following four propositions are equivalent to the other:

1. There are some \( \beta_1, C_1 > 0 \) such that the estimate (2) holds for any \( u \in H^m_{\beta}(\Omega). \)
2. There are some \( \beta_2, C_2 > 0, \) such that the estimate

\[
\text{Re}((A + \beta)f, f)_{L^2(\Omega \times T)} \geq C_2 \| f \|_{H^{m-1/2}(\Omega \times T)}
\]

holds for any \( f \) in \( H^m_{\beta}(\Omega \times T). \)
3. There are some constants \( \beta_3, C_3 > 0 \) such that the estimate

\[
\sum_{p,q=m-j}^{m-1} (H_{pq}(\beta) \varphi_q, \varphi_p)_{L^2(\Gamma \times T)} \geq C_3 \sum_{p=m-j}^{m-1} \| \varphi_p \|_{H^{m-1/2}(\Gamma)}^2
\]

holds for any \( p, q \in (m-j, \cdots, m-1) \in H^m_{\beta}(\Omega \times T). \)
4. There are some constants \( \gamma, \beta_4, C_4 > 0 \) such that the estimate

\[
\sum_{p,q=m-j}^{m-1} (H_{pq}(\gamma) \varphi_q, \varphi_p)_{L^2(\Gamma \times T)} \geq \beta_4 \sum_{p=m-j}^{m-1} \| \varphi_p \|_{H^{m-1/2}(\Gamma \times T)}^2
\]

holds for any \( \varphi_{m-j}, \varphi_{m-j+1}, \cdots, \varphi_{m-1} \in H^m_{\beta}(\Omega \times T). \)

**Remark 1.** In the case \( 0 < s < 1/2 \) the estimate holds with some \( \beta, C > 0, \) if and only if, with some \( \gamma, \beta, C > 0, \) the estimate

\[
\sum_{p,q=m-j}^{m-1} (H_{pq}(\gamma) \varphi_q, \varphi_p)_{L^2(\Gamma \times T)} \geq C \sum_{p=m-j}^{m-1} \| \varphi_p \|_{H^{m-1/2}(\Gamma \times T)}^2
\]

holds for any \( \varphi_{m-j}, \cdots, \varphi_{m-1} \in H^m_{\beta}(\Omega \times T). \)

We consider pseudo-differential operators \( \tilde{H}_{pq}(\gamma), m-j \leq p, q \leq m-1, \) of order \( 2m - 1 \) defined on \( \Gamma \times T \) and satisfying the property (iv) of Theorem 1.

The property (iv) of Theorem 1 can be localized.

**Theorem 2.** Assume that there exists a family of finite number of real functions \( \{ \mu_k(x) \}_{k=1}^{\infty} \) in \( \mathcal{D} \) (\( \Gamma \times T \)) satisfying

1. \( \sum \mu_k(x, s) s^k = 1, \)
2. for any \( \phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1} \in \mathcal{D} \) (\( \Gamma \times T \)) and for any \( k \) the following estimate holds:

\[
\sum_{p,q=m-j}^{m-1} (H_{pq}(\gamma) \varphi_q, \varphi_p)_{L^2(\Gamma \times T)} \geq C \sum_{p=m-j}^{m-1} \| \varphi_p \|_{H^{m-1/2}(\Gamma \times T)}^2
\]
Then for any \( \phi_{m-j}, \phi_{m-j+1}, \ldots, \phi_{m-1} \in \mathcal{D}(\Gamma \times T) \) the estimate (13) holds with some \( \beta, C_4, \gamma > 0 \).

Let \( \Omega \) be any open set (not necessarily connected) in \( \mathbb{R}^n \). Let \( Q_{rs}, \quad m-j \leq r, s \leq m-1, \) be pseudo-differential operators of order 1 defined in \( \Omega \). \( q_{rs}(x, \xi) = \sum_{j=0}^\infty q_{rs}^j(x, \xi) \) denote the symbol of \( Q_{rs} \). We assume the matrix \( (q_{rs}^j(x, \xi))_{rs} \) of the principal symbols of \( Q_{rs} \) is Hermitian. Then we have

**Theorem 3.** The following two properties are equivalent:

(i) For any compact set \( K \) in \( \Omega \), there are constants \( C_0 \) and \( C_1 > 0 \) such that, for any \( \phi_{m-j}, \phi_{m-j+1}, \ldots, \phi_{m-1} \in \mathcal{D}(K) \),

\[
\sum_{r,s=m-j}^{m-1} \left( Q_{rs} \phi_r \phi_s L^2(\Omega) + C_1 \sum_{r=m-j}^{m-1} \| \phi_r \|_{H^{-1/2}(\Omega)}^2 \right) \geq C_0 \sum_{r=m-j}^{m-1} \| \phi_r \|_{H^0(\Omega)}^2.
\]

(ii) For any compact set \( K \) in \( \Omega \), there exist constant \( C > 0 \), integer \( N > 0 \) and a function \( \varepsilon(\xi) \) with \( \varepsilon(\xi) \to 0 \) when \( |\xi| \to \infty \) such that, for any \( x \in K, \phi_{m-j}, \ldots, \phi_{m-1} \in \mathcal{D}(\mathbb{R}^n) \),

\[
\sum_{r,s=m-j}^{m-1} \sum_{|\xi|+|\beta| \leq 2} \left| \varepsilon(\xi) \right| \left| q_{rs}^j(x, \xi) \int_{\mathbb{R}^n} (iD_y)^\beta \phi_s(y) (-iy)^\beta \psi_r(y) dy \right|^2
\]

\[
+ \sum_{r=m-j}^{m-1} \int_{\mathbb{R}^n} |D_y \phi_r(y)|^2 dy
\]

\[
\geq C \sum_{r=m-j}^{m-1} \left| \psi_r(y) \right|^2 dy,
\]

where \( q_{rs}^j(x, \xi) = D_x^r D_y^s q(x, \xi) \).

**Remark 2.** The estimate (14) holds for any \( \phi_{m-j}, \ldots, \phi_{m-1} \in H^{m-1/2}(\Gamma) \) if and only if the matrix defined by the principal symbols \( \sigma_{m-j}(H_{pq}(B))(x, \xi) \) is uniformly positive definite. Thus we can prove the result in [3] without the assumption that \( \sigma_{2m}(A)(x, \xi) \) is real.

To prove Theorem 3 we use the following theorem which is interesting in itself.

**Theorem 4.** Let \( K \) be any compact set in an open set \( \Omega \) in \( \mathbb{R}^n \) and let \( P \) be a pseudo-differential operator of order \( \rho \) defined on \( \Omega \), whose symbol is denoted by \( p(x, \xi) \). Assume \( \varphi \in \mathcal{D}(\Omega) \) is identically 1 in some neighbourhood of \( K \). Then for any \( N > 0 \), there is a constant \( C > 0 \) such that for any \( x \in K, \xi \in \mathbb{R}^n \) with \( |\xi| \geq 1 \), and \( \phi, \varphi \in \mathcal{D}(\mathbb{R}^n) \),

\( \star \) During the preparation of this article the author had a chance to know that A. P. Calderón also had obtained, independently, a result similar to Theorem 4 in a little stronger form.
\[ |\xi|^{n/2} \int \phi(y) (P \varphi v_2)(y) \varphi v_2(y) dy \]

\[ - \sum_{|\alpha|, |\beta| < \alpha ! \beta !} c_{\alpha, \beta}(x, \xi) \int (y \zeta)^{\alpha} \varphi(y) (-iy)^{\beta} \varphi(y) dy \]

\[ \leq C |\xi|^{-n/2} |\varphi(y)\|_{H^{N/2}} (1 + |y|)^{N} \| \phi \|_{H^{N/2}(\mathbb{R}^n)}, \]

where \( v_1(y) = \psi((y - x) |\xi|^{1/2}) e^{iy \cdot \zeta} \) and \( v_2(y) = \phi((y - x) |\xi|^{1/2}) e^{iy \cdot \zeta} \).

Proofs of Theorems 3 and 4 are omitted here. They are similar to the discussion in [4].

References


