53. On Ranked Spaces and Linearity. II

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(Comm. by Kinjirō Kunugi, M. J. A., April 12, 1969)

In this note we shall give a definition of linear ranked spaces, axioms of which are weaker than those given in [2]. Sometimes this definition is more convenient to use, in particular, to study for the notions connected with fundamental sequences of neighbourhoods. Hereafter we shall treat only ranked spaces with indicator $\omega_0[1]$. Throughout this note, $x, y, \cdots$ will denote points of a ranked space, $\mathcal{N}_n(x)$ the system of neighbourhoods of $x$ with rank $n$, $\{u_n(x)\}, \{v_n(x)\}, \cdots$ fundamental sequences of neighbourhoods with respect to $x$.

§ 1. Definition of linear ranked spaces. Let $E$ be a ranked space, and also a linear space over real or complex field. We call $E$ a linear ranked space, if linear operations in $E$ are continuous in the following sense:

(I) For any $\{u_n(x)\}$ and $\{v_n(y)\}$, there is a $\{w_n(x+y)\}$ such that $u_n(x) + v_n(y) \subseteq w_n(x+y)$.

(II) For any $\{u_n(x)\}$ and $\{\lambda_n\}$ with $\lim \lambda_n = \lambda$, there is a $\{v_n(\lambda x)\}$ such that $\lambda_n u_n(x) \subseteq v_n(\lambda x)$.

(I) implies the continuity of addition; if $\{\lim x_n\} \ni x$ and $\{\lim y_n\} \ni y$, then $\{\lim (x_n + y_n)\} \ni x + y$, and (II), the continuity of scalar multiplication; if $\{\lim x_n\} \ni x$ and $\lim \lambda_n = \lambda$, then $\{\lim \lambda_n x_n\} \ni \lambda x$.

§ 2. The neighbourhoods of zero. Let $E$ be a linear ranked space. We will denote the system of neighbourhoods of 0 with rank $n$ by $\mathcal{N}_n$, and fundamental sequences with respect to 0 by $\{U_n\}, \{V_n\}, \cdots$. Obviously $\mathcal{N}_n$ satisfies the axioms (A), (B), (a), (b) in [2].

Furthermore, from (I), (II), we get following properties.

(RL1) For any $\{U_n\}$ and $\{V_n\}$, there is a $\{W_n\}$ such that $U_n + V_n \subseteq W_n$.

(RL2) For any $\{U_n\}$ and $\lambda$, there is a $\{V_n\}$ such that $\lambda U_n \subseteq V_n$.

(i) For any $x$ and $\{\lambda_n\}$ with $\lim \lambda_n = 0$, there is a $\{V_n\}$ such that $\lambda_n x \in V_n$.

(ii) For any $\{U_n\}$ and $\lambda_n$ with $\lim \lambda_n = 0$, there is a $\{V_n\}$ such that $\lambda_n U_n \subseteq V_n$.

(RL3) Let $x$ be any point in $E$. For any $\{U_n\}$ there is a $\{v_n(x)\}$ such that $x + U_n \subseteq v_n(x)$, and, conversely, for any $\{u_n(x)\}$ there is a $\{V_n\}$ such that $u_n(x) \subseteq x + V_n$. 

Proof. (RL₁), (RL₂) (i), (iii) are immediate consequences of (I), (II), respectively, putting \( x = y = 0, \lambda_n = \lambda, \) or \( \lambda = 0. \) As for (RL₂) (ii), taking some \( \{u_n(x)\} \) and applying (II) for \( \lambda = 0, \) there is a \( \{V_n\} \) such that \( \lambda_n u_n(x) \subseteq V_n. \) Since \( x \in u_n(x), \) we have \( \lambda_n x \in V_n. \) Now, we shall show (RL₃). Take any \( \{U_n\} \) and some \( \{u_n(x)\} \). From (I), there is a \( \{v_n(x)\} \) such that \( u_n(x) + U_n \subseteq v_n(x), \) and therefore \( x + U_n \subseteq v_n(x). \) Conversely, for any \( \{u_n(x)\}, \) taking some \( \{v_n(-x)\}, \) there is a \( \{V_n\} \) such that \( u_n(x) + v_n(-x) \subseteq V_n, \) and therefore \( -x + u_n(x) \subseteq V_n, \) i.e., \( u_n(x) \subseteq x + V_n. \) Q.E.D.

The three conditions above are not only necessary, but sufficient for a linear space which is also a ranked space to be a linear ranked space. In other words, (I), (II) follow from (RL₁), (RL₂), (RL₃). It is clear that (RL₃) (iii) can be omitted if every \( V \) in \( \mathcal{R} \) is circled.

Proof. (I) Let \( \{u_n(x)\}, \{v_n(y)\} \) be any fundamental sequences. From (RL₃), there are \( \{U_n\}, \{V_n\} \) such that \( u_n(x) \subseteq x + U_n, v_n(y) \subseteq y + V_n. \) Applying (RL₁), there is a \( \{W_n\} \) such that \( U_n + V_n \subseteq W_n. \) From (RL₃) again, there is a \( \{w_n(x+y)\} \) such that \( x + y + W_n \subseteq w_n(x+y). \) Thus, \( u_n(x) + v_n(y) \subseteq (x + U_n) + (y + V_n) \subseteq x + y + W_n \subseteq w_n(x+y). \)

(II) Take any \( \{u_n(x)\} \) and \( \{\lambda_n\} \) with \( \lim \lambda_n = \lambda. \) From (RL₃), there is a \( \{U_n\} \) such that \( u_n(x) \subseteq x + U_n. \) Putting \( \mu_n = \lambda_n - \lambda, \) we have \( \lim \mu_n = 0, \) and therefore by (RL₃) (i), (ii), (iii), respectively, there are \( \{V^1_n\}, \{V^2_n\}, \{V^3_n\} \) such that \( \lambda U_n \subseteq V^1_n, \mu_n x \in V^2_n, \mu_n U_n \subseteq V^3_n. \) Then, \( \lambda_n u_n(x) \subseteq (\lambda + \mu_n)(x + U_n) \subseteq \lambda x + \lambda U_n + \mu_n x + \mu_n U_n \subseteq \lambda x + V^1_n + V^2_n + V^3_n. \) Applying (RL₁), there is a \( \{V_n\} \) such that \( V^1_n + V^2_n + V^3_n \subseteq V_n. \) Finally, from (RL₃) again, there is a \( \{v_n(\lambda x)\} \) such that \( \lambda x + V_n \subseteq v_n(\lambda x). \) Thus, we have a \( \{v_n(\lambda x)\} \) such that \( \lambda_n u_n(x) \subseteq v_n(\lambda x). \) Q.E.D.

In many important examples it seems natural to take \( \{x + V; V \in \mathcal{R}_n\} \) as \( \mathcal{R}_n(x). \) If we do so, (RL₂) is automatically fulfilled. Thus, when in a linear space \( E, \) families \( \mathcal{R}_n \) are given and satisfy axioms (A), (B), (a), (b), (RL₁), (RL₂), \( E \) becomes a linear ranked space taking \( \mathcal{R}_n(x) \) as above.

It is easily seen that axioms (1), (2), (3) in [2] are sufficient conditions for (RL₁) (RL₂) (i), (ii), respectively, when every \( V \) in \( \mathcal{R} \) is circled.

§ 3. Examples. As remarked above, all examples in [2] are linear ranked spaces. We shall give an example which is not a linear ranked space in earlier sense.

Let \( \Phi \) be the union space of countably normed spaces \( \Phi^{(p)} (p = 1, 2, \ldots) \) [5], i.e. \( \Phi = \bigcup_{p=1}^{\infty} \Phi^{(p)}, \) where

(1) \( \Phi^{(p)} \subseteq \Phi^{(p+1)} \)

(2) the systems \( \{\| \cdot \|_{n=1,2,\ldots}^{(p)}\} \) and \( \{\| \cdot \|_{n=1,2,\ldots}^{(p+1)}\} \) are equivalent in \( \Phi^{(p)}. \)

1) \( \| \cdot \|_{n=1,2,\ldots}^{(p)} \) will denote the system of norms in \( \Phi^{(p)}. \) We assume that \( \| \cdot \|_{n=1,2,\ldots}^{(p)} \leq \| \cdot \|_{n=1,2,\ldots}^{(p+1)} \leq \ldots. \)
Put $v(n, p ; 0) = \{ \varphi \in \Phi^{(p)} ; \| \varphi \|^{(p)}_n < \frac{1}{n} \}, \mathcal{B}_n = \{ v(n, p ; 0) ; p = 1, 2, \ldots \}$ for $n \geq 1$, and $\mathcal{B} = \{ \Phi \}$. It is clear that, if $m \geq n$, then $v(m, p ; 0) \subseteq v(n, p ; 0)$. Moreover it can be shown that, if $v(m, p ; 0) \subseteq v(n, q ; 0)$, then necessarily $p \leq q$.

Obviously axioms (A), (a), (b) are satisfied. To prove (B), we take any $U = v(m, p ; 0)$ and $V = v(n, q ; 0)$. We may assume $p \leq q$. From the hypothesis (2), there are $m'$ and $M$ such that $\| \varphi \|^{(q)} \leq M \| \varphi \|^{(p)}_n$ for $\varphi \in \Phi^{(p)}$. Taking sufficient large $m''$, we have $m'' \geq m$ and $\| \varphi \|^{(q)}_n \leq \frac{m''}{n} \| \varphi \|^{(p)}_n$ for $\varphi \in \Phi^{(p)}$. Now, $W = v(m'', p ; 0) \subseteq U \cap V$.

Thus, taking $\mathcal{B}_n(\varphi) = \{ \varphi + V ; V \in \mathcal{B}_n \}, \Phi$ becomes a ranked space. We shall show that convergence of sequences in $\Phi$ is equivalent to usual one; we have \( \lim \varphi_i \neq 0 \) if and only if all $\varphi_i$ belong to $\Phi^{(p)}$ for some fixed $p$, and $\{ \varphi_i \}$ converges to 0 in $\Phi^{(p)}$, i.e. $\lim \| \varphi_i \|^{(p)}_n = 0$ for each $n$. If $\lim \varphi_i \neq 0$, there is a $\{ U_i \}$ such that $\varphi_i \in U_i$. Let $U = v(n, p ; 0)$. Since $U_i \supseteq U_{i+1}$, we have $p_i \geq p_{i+1}$, and therefore, for some $i_0$,

\[
p_i = p_0 = \min p_i \quad \text{when} \quad i \geq i_0.\]

Since $n \uparrow \infty$, we have $\| \varphi_i \|^{(p)}_n \rightarrow 0$ for each $n$. Thus $\varphi_i$ belongs to $\Phi^{(p)}$ for $i \geq i_0$ and converges to 0 in $\Phi^{(p)}$. From the hypothesis (2), $\{ \varphi_i \} \subseteq \mathcal{B}_n(\varphi)$ converges to 0 in $\Phi^{(p)}$, too. Obviously all $\varphi_i$ belong to $\Phi^{(p)}$, and converges to 0 in $\Phi^{(p)}$.

On the other hand, if for some fixed $p$, $\varphi_i \in \Phi^{(p)}$ and $\varphi_i$ converges to 0 in $\Phi^{(p)}$, we can choose an increasing sequence of positive integers, $\{ i_n \}$, such that $\| \varphi_i \|^{(p)}_n < \frac{1}{n}$ for $i \geq i_n$. Putting $U_i = v(n, p ; 0)$ for $i$ with $i_n \leq i < i_{n+1}$, and $U_i = \Phi$ for $i < i_n$, we get a fundamental sequence $\{ U_i \}$ such that $\varphi_i \in U_i$.

Now, we shall prove (RL1). Let $\{ U_i \}, \{ V_i \}$ be fundamental sequences, where $U_i = v(n, p ; 0), V_i = v(m, q ; 0)$. We must make a $\{ W_i \}$ such that $U_i + V_i \subseteq W_i$. As shown before, there are $p, q, N$ such that $p_i = p$ and $q_i = q$ for $i \geq N$. We can assume $N = 1$. If $p = q$, putting $W_i = v(n, p ; 0)$, where $n \downarrow = \min \left( \left[ \frac{l}{2} \right], \left[ \frac{m}{2} \right] \right)$, we have $U_i + V_i \subseteq W_i$.

When $p \neq q$, we may suppose $p > q$. Since systems $\{ \| \varphi \|^{(p)} \}$ and $\{ \| \varphi \|^{(q)} \}$ are equivalent in $\Phi^{(q)}$ and $m \uparrow \infty$ there exist $i$, and $C$ such that $\frac{m}{2} \| \varphi \|^{(q)}_i \geq \| \varphi \|^{(p)}_i$ for any $\varphi \in \Phi^{(q)}$, and for $i \geq i$. Thus, when $i \geq i_0$, $\| \varphi \|^{(p)}_i < \frac{1}{2}$ for any $\varphi \in \mathcal{B}_i$. Repeating this process, we obtain an increasing sequence $\{ i_i \}$ such that, when $i \geq i_i$, $\| \varphi \|^{(p)}_i < \frac{1}{2^i}$ for $\varphi \in \mathcal{B}_i$. Moreover, we can assume $l_i \geq 2i$ for $i \geq i_i$. Now, putting $W_i = v(n, p ; 0)$ for $i$ with
\( i_1 \leq i < i_{i+1} \), and \( W_i = \emptyset \) for \( i < i_1 \), we have \( U_i + V_i \subseteq W_i \).

Clearly every \( V \) in \( \mathcal{B} \) is circled. The axioms (2) and (3) in [2] hold, putting \( \psi(\lambda, \mu) = \left[ \frac{\lambda}{\mu} \right] \). Hence (RLn) is fulfilled.

Finally we remark that \( \Phi \) may not satisfy the axiom (1) in [2]; suppose that the inequalities \( \| \varphi \|_p^{(p)} \geq p \cdot \| \varphi \|_p \) \((p = 1, 2, \ldots)\) hold for every \( \varphi \in \Phi^{(p)} \). Then, for any \( l \) and \( m \) there are \( U \) and \( V \), respectively in \( \mathcal{B} \) and \( \mathcal{B}_m \) such that no \( W \) in \( \mathcal{B}_n \), \( n \geq 1 \), can include \( U + V \). In fact, let \( U = v(l, 1; 0) \), \( V = v(m, l + 1; 0) \), and suppose \( U + V \subseteq W \), where \( W = v(n, p; 0) \), \( n \geq 1 \). Since \( V \subseteq W \), \( p \geq l + 1 \). For any \( \varphi \in W \), \( \| \varphi \|_p^{(p)} < \frac{1}{n} \), a fortiori \( \| \varphi \|_p < 1 \). If we take a \( \varphi \in \Phi^{(p)} \) such that \( \| \varphi \|_p^{(p)} = 1 \), clearly \( \varphi \in W \). On the other hand, \( 1 = \| \varphi \|_p^{(p)} \geq p \cdot \| \varphi \|_p^{(p)} \geq (l + 1) \cdot \| \varphi \|_p^{(p)} \), therefore \( \| \varphi \|_p^{(p)} \leq \frac{1}{l + 1} < \frac{1}{l} \), i.e. \( \varphi \in U \). Hence \( U \not\subseteq W \). This contradicts the fact that \( U + V \subseteq W \).

§ 4. Bounded sets in linear ranked spaces. We already gave a definition of bounded sets in linear ranked spaces in earlier sense in [3], and another definition in [4]. Now, let \( E \) be a linear ranked space in new sense. We use Definition 2 in [4]: A subset \( B \) in \( E \) is called bounded if there is a fundamental sequence \( \{ V_n \} \), any member of which absorbs \( B \). The study for bounded sets in [4] can be applied to our case. For example, from (RLn), it follows that any finite union and finite sum of bounded sets are also bounded, from (RLn) (i), that any scalar multiple of bounded set is bounded, and from (RLn) (ii), that any one point set is bounded.

We give a sufficient condition for the property that every convergent sequence is bounded. It is as follows: For any \( \{ U_n \} \) there is a \( \{ V_n \} \) such that every \( V_n \) is circled and for some \( N \), \( \bigcup \lambda V_n = \bigcup \lambda V_N \) for \( n \geq N \). The proof is trivial and omitted. The union of countably normed spaces \( \Phi \) in § 3 evidently satisfies this condition.

Now, we shall show that, in \( \Phi \), boundedness is equivalent to usual one; \( B \) is bounded in our sense, if and only if \( B \) is included in some \( \Phi^{(p)} \) and \( \sup_{\varphi \in B} \| \varphi \|_n^{(p)} < \infty \) for each \( n \). In fact, suppose that, for some \( \{ V_i \} \), where \( V_i = v(n_i, p_i; 0) \), every \( V_i \) absorbs \( B \), and let \( p = \min_i p_i \). Clearly, \( B \subseteq \Phi^{(p)} \) and \( \sup_{\varphi \in B} \| \varphi \|_n^{(p)} < \infty \) for each \( n \). On the other hand if \( B \subseteq \Phi^{(p)} \) and \( \sup_{\varphi \in B} \| \varphi \|_n^{(p)} < \infty \) for each \( n \), then putting \( U_n = v(n, p; 0) \), we get a \( \{ U_n \} \) any member of which absorbs \( B \).

2) This is possible, when we omit some finite members of \( \{ \| \varphi \|_n^{(p)} \} \), and multiply by some positive numbers. In each \( \Phi^{(p)} \), the new system of norms is equivalent to initial one, and therefore convergence of sequence in \( \Phi \) is unaltered.
Finally we remark that first definition of boundedness in [3] (Definition 1 in [4]) is not always equivalent to usual one. To prove this, we put \( \Phi = \Omega \), \( \Phi^{(p)} = \Omega_p = \{ \varphi \in \Omega ; \text{car} \varphi \subseteq [-p, p] \} \) and define the systems of norms as follows. Let \( \| \varphi \|_n = \max_{0 \leq j \leq n-1} \sup_x |\varphi^{(j)}(x)| \). In \( \Phi^{(p)} \), let \( |\varphi|_p = \sup_{|\varphi| \leq 1} |\varphi(\psi)|_p \), \( |\varphi|_{p+1} = \sup_{|\varphi| \leq 1} |\varphi(\psi)|_{p+1} \), \( |\varphi|_{p+2} = \|\varphi\|_n (n=1, 2, \ldots) \). Obviously two systems \( \{ \| \cdot \|_n \} \) and \( \{ | \cdot |_n \} \) are equivalent in \( \Phi^{(p)} \), and therefore convergence of sequences in \( \Phi \) coincides with usual one. In this space \( \Phi \), \( V = v(2, 1; 0) = \{ \varphi \in \Omega_1 ; |\varphi|_{[1]} = \sup_x |\varphi(x)| < \frac{1}{2} \} \) is bounded by Definition 1; for any \( n \) there is a \( U \) in \( \Omega_n \) which absorbs \( V \). In fact, let \( U = v(n, n-1; 0) \). Since \( |\varphi|_{n+1} = \|\varphi\|_1 \), \( U = \{ \varphi \in \Omega_{n-1} ; \sup_x |\varphi(x)| < \frac{1}{n} \} \). Hence \( \frac{2}{n} V \subseteq U \). It is clear that this set is not bounded in usual sense.

References


\[ \varphi(\psi) = \int_{-\infty}^{\infty} \varphi(x) \psi(x) dx \] where \( \varphi, \psi \in \Omega \).