54. A Remark on the Theorem of Bishop

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1. On normality of a family of pure-dimensional analytic sets in a domain of $C^n$, the following theorem of Oka [4] is well-known.

**Theorem of Oka.** Let $F$ be a family of pure-dimensional analytic sets in a domain of $C^n$. Then $F$ is analytically normal if and only if the volumes of elements of $F$ are locally uniformly bounded.

This theorem was proved by T. Nishino [3] in the case of two variables. The proof of this theorem in the case of $n$ variables was given in our former paper (Watanabe [6]).

On the other hand, the concept of geometric convergence was introduced by E. Bishop as follows.

Let $\{S_j\}$ be a sequence of closed subsets in a domain of $C^n$. It is said that $\{S_j\}$ converges geometrically to a closed set $S$ if for any compact set $K$, $\{S_j \cap K\}$ is a convergent sequence in $\operatorname{Comp}(K)^{\mathbb{N}}$ and $S = \bigcup K \lim (S_j \cap K)$ where $K$ ranges over the compact sets. Further Bishop [1] proved the following.

**Theorem of Bishop.** Let $\{S_j\}$ be a sequence of purely 2-dimensional analytic sets in a domain $D$ of $C^n$. Suppose that $\{S_j\}$ converges geometrically to a closed set $S$ in $D$. If the volumes of $S_j$ are uniformly bounded, then $S$ is also an analytic set in $D$.

We shall prove that in the above theorem of Bishop, $S$ is also purely 2-dimensional if $S$ is not empty.

2. Let $D = \Delta \times \{|w| < R\}$ be a domain of $C^{n+1}$, where $\Delta$ is a domain of $(z_1, \ldots, z_n)$-space $C^n(z)$. Then the following proposition is well-known (for example, Fujita [2]).

**Proposition.** Let $S$ be a purely 2-dimensional analytic set in $D$. Assume that $S$ is contained in $\Delta \times \{|w| < R_0\}$ for some positive number $R_0 < R$. Then the projection of $S$ on $\Delta$ is also purely 2-dimensional analytic set in $\Delta$.

It follows from this:

**Corollary.** Let $D = \Delta \times \{|w_1| < R\} \times \cdots \times \{|w_p| < R\}$ be a domain of $C^{n+p}$ and $S$ be a purely 2-dimensional analytic set in $D$. If $S$ is contained in $\Delta \times \{|w_1| < R_0\} \times \cdots \times \{|w_p| < R_0\}$ for some positive number $R_0 < R$, then $\mathfrak{A} = (S, \pi, \Delta)$ is an analytic cover, where $\pi$ is a projection.

1) For a definition of $\operatorname{Comp}(K)$, see [5].
Now let us prove the following

**Theorem.** Let \( \{S_i\} \) be a sequence of purely \( \lambda \)-dimensional analytic sets in a domain of \( \mathbb{C}^n \). If \( \{S_i\} \) converges geometrically to a non-empty analytic set \( S \), then the local dimension \( \dim_p S \) at each point \( p \in S \) is at least \( \lambda \).

**Proof.** Suppose that there holds \( \dim_p S = k < \lambda \) for some point \( p \in S \). For simplicity we may assume that \( p \) is the origin. After a suitable change of the coordinate system, we can choose \( (n-k) \) pseudo-polynomials \( P_{k+1}(z'; \zeta), P_{k+2}(z'; \zeta), \ldots, P_n(z'; \zeta) \), whose coefficients are holomorphic functions of \( z' = (z_1, \ldots, z_k) \), and positive numbers \( \varepsilon_i \) \( (i=1, 2, \ldots, n) \) such that

(i) \( S \cap \Omega = \{ (z', z_{k+1}, \ldots, z_n); P_1(z'; z_1) = 0, 1 = k+1, \ldots, n \} \) for a neighbourhood \( \Omega \) of the origin.

(ii) the roots of \( P_l(z'; \zeta) = 0 \) are all contained in the disc \( |\zeta| < \varepsilon_j \) \( (l=k+1, \ldots, n) \) for \( z' = (z_1, \ldots, z_k) \) with \( |z_j| \leq \varepsilon_j \) \( (j=1, 2, \ldots, k) \). For sufficiently small positive numbers \( r, \rho \), the polydisc \( \Omega = \Delta \times \{ |z_{k+1}| < \rho \} \times \ldots \times \{ |z_n| < \rho \} \) is a relatively compact subset of \( U \), where \( \Delta = \{ z'; |z_j| < r, j=1, 2, \ldots, k \} \).

We may assume that for \( z' \in \Delta \), the roots of \( P_l(z'; \zeta) = 0 \) are all contained in the disc \( |\zeta| < \frac{\rho}{2} \). Then \( S^* \) is an analytic set in \( \Omega \) and \( S^* \cap \Omega \subset \Delta \times \{ |z_{k+1}| < \rho' \} \times \ldots \times \{ |z_n| < \rho' \} \). Since \( \{S_i\} \) converges geometrically to \( S \), it follows that \( \lim (S_i \cap \Omega) \subset S \cap \Omega \), and it is easily seen that for a positive number \( \rho'' \) \( (\rho' < \rho'' < \rho') \) there is a positive integer \( \nu_0 \) such that \( S_i \cap \Omega \) is contained in \( \Delta \times \{ |z_{k+1}| < \rho'' \} \times \ldots \times \{ |z_n| < \rho'' \} \) for \( \nu \geq \nu_0 \). On the other hand, denoting \( \Delta \times \{ |z_{k+1}| < \rho \} \times \ldots \times \{ |z_n| < \rho \} \) by \( \Delta' \), we have \( \Omega = \Delta' \times \{ |z_{k+1}| < \rho' \} \times \ldots \times \{ |z_n| < \rho' \} \) and so \( S_i \cap \Omega \subset \Delta' \times \{ |z_{k+1}| < \rho'' \} \times \ldots \times \{ |z_n| < \rho'' \} \). But from the corollary of Proposition, the projection of \( S_i \cap \Omega \) on the \( (z_1, z_2, \ldots, z_k) \)-space is \( \Delta \). This contradicts the fact that \( S_i \cap \Omega \) is contained in \( \Delta \times \{ |z_{k+1}| < \rho'' \} \times \ldots \times \{ |z_n| < \rho'' \} \).

Q.E.D.

On the other hand, from the estimation of the Hausdorff measure and the relation between the volume and the Hausdorff measure, the dimension of \( S \) is at most \( \lambda \) if a sequence of purely \( \lambda \)-dimensional analytic sets converges geometrically to an analytic set \( S \) and if the \( 2\lambda \)-dimensional volumes of \( S_i \) are uniformly bounded ([5]). Therefore we have from our theorem, the following

**Corollary.** In the theorem of Bishop, if the limit set \( S \) is not empty, then \( S \) is also purely \( \lambda \)-dimensional.

3. Here we shall give some properties of geometric and analytic convergence.

Let \( \{S_i\} \) be a sequence of closed sets in a domain \( D \) of \( \mathbb{C}^n \). Sup-
pose that \( \{S_v\} \) converges geometrically to a closed set \( S \) in \( D \). Since
\[
\lim(S_v \cap K_p) \subseteq \lim(S_v \cap K_{p+1}),
\]
we have \( S \subseteq \lim \lim(S_v \cap K_p) \) for every sequence \( \{K_p\} \) of compact sets such that \( K_p \subseteq K_{p+1}, \ldots \), and \( D = \bigcup K_p \). On the other hand, we have \( S \supseteq \lim \lim(S_v \cap K_p) \) from the very definition, and hence \( S = \lim \lim(S_v \cap K_p) \). The converse is not necessarily true as the following example shows.

Example. Let \( D = \{(z_1, z_2) \in \mathbb{C}^2; |z_1-1| < 1, |z_2| < 1\} \) be a domain of \( \mathbb{C}^2 \) and \( \{S_n\} \) be a sequence of analytic sets such that \( S_1 = \{(z_1, z_2) \in D; z_1 = \frac{1}{2}\}, S_2 = \{(z_1, z_2) \in D; z_1 = 1 + \frac{1}{2}\}, \ldots, S_{2n} = \{(z_1, z_2) \in D; z_1 = 1 + \frac{1}{2^n}\}, \ldots \). Then since \( S_{2n} \cap K = 0 \) for the compact set \( K = \{(z_1, z_2) \in D; |z_1| < 1, |z_2| < 1\} \), \( S_{2n+1} \cap K = \emptyset \) for the compact set \( K = \{(z_1, z_2) \in D; |z_1| < \frac{5}{4}, |z_2| < \frac{1}{4}\} \), \( \{S_n\} \) does not converge geometrically. But it is obvious that \( \lim \lim(S_v \cap K_p) = \{(z_1, z_2) \in D; z_1 = 1\} \) for every exhaustion of \( D \) by compact sets \( K_p \). However if \( \lim(S_v \cap K_p) \) exists for sufficiently large \( p \), then there is a subsequence \( \{S_v\} \) of \( \{S_v\} \) which converges geometrically to a closed set \( S^* \) and \( \lim(S_v \cap K_p) = \lim(S_v \cap K_p) \) for sufficiently large \( p \), and hence we have \( S^* = \lim \lim(S_v \cap K_p) = \lim \lim(S_v \cap K_p) \) for sufficiently large \( p \).

Summing up the above result, we have

Proposition 1. If \( \{S_v\} \) converges geometrically to a closed set \( S \) in \( D \), then it holds that \( S = \lim \lim(S_v \cap K_p) \) for every sequence \( \{K_p\} \) of compact sets such that \( K_p \subseteq K_{p+1}, \ldots \), and \( D = \bigcup K_p \). Further if \( S = \lim \lim(S_v \cap K_p) \) exists, then \( S \) is closed and there is a subsequence of \( \{S_v\} \) which converges geometrically to \( S \).

Next we consider the case of analytic convergence of a sequence of pure-dimensional analytic sets.

Suppose that a sequence \( \{S_v\} \) of pure-dimensional analytic sets in a domain \( D \) of \( \mathbb{C}^n \) converges analytically to \( S \).\(^2\) We shall show that \( S \cap K = \lim(S_v \cap K) \) for a compact set \( K \) such that \( S \cap K \neq \emptyset \).

Let \( S^{(\varepsilon)} = \bigcup_{\varepsilon > 0} \{z; \rho(z, z') < \varepsilon\} \), where \( \rho(z, z') \) means the Euclid distance between \( z \) and \( z' \). If \( S_v \cap K - S^{(\varepsilon)} \cap K \neq \emptyset \) for a sequence of positive integers \( \nu_1 < \nu_2 < \ldots \), then we can choose a sequence of points \( p_{\nu_j} \in S_{\nu_j} \cap K - S^{(\varepsilon)} \cap K \). Since \( K \) is compact it may be assumed that \( p_{\nu_j} \to p \). From the assumption \( p \) is not contained in \( S \). On the other

\(^2\) For a definition of analytic convergence, see [2], [6].
hand, from the definition of analytic convergence there are a neigh-
bourhood \( U \) of \( p \) and holomorphic functions \( f_k^{(i)} \), \( k=1, 2, \ldots, l \) in \( U \) such that \( S_i \cap U = \{ z \in U ; f_k^{(i)}(z) = 0, \ k=1, 2, \ldots, l \} \).

Moreover, since \( f_k^{(i)}(z) \) converges uniformly to a holomorphic function \( f^{(i)}(z) \) in \( U \), \( f_k^{(i)}(z) \) also converges uniformly to \( f^{(i)}(z) \) in \( U \). We have \( |f_k^{(i)}(p)| = \delta_k > 0 \) since \( p \) is not contained in \( S \). But since \( f_k^{(i)}(z) \) converges uniformly to \( f^{(i)}(z) \), it holds \( |f_k^{(i)}(p)| \leq |f_k^{(i)}(p) - f_k^{(i)}(p_v)| + |f_k^{(i)}(p_v) - f^{(i)}(p_v)| < \delta_k \) for sufficiently large \( j \). This is a contradiction and hence we have \( S_i \cap K - S^{(i)} \cap K = \emptyset \) for sufficiently large \( j \).

Thus there is a positive integer \( \nu \) depending only on \( \varepsilon \) such that \( S_i \cap K \subseteq S^{(i)} \cap K \) for \( \nu \geq \nu_0 \). This means that \( \{ S_i \cap K \} \) converges to \( S \cap K \) in \( \text{Comp}(K) \). Thus we have

**Proposition 2.** If a sequence \( \{ S_i \} \) of pure-dimensional analytic sets converges analytically to \( S \), then it holds that \( S = \lim \lim (S_i \cap K) \) for every sequence \( \{ K_i \} \) of compact sets such that \( K_i \subseteq K_{i+1} \ldots \), and \( D = \bigcup_i K_i \).

**Remark.** Even if \( \lim (S_i \cap K) \) exists, the sheet numbers of \( S_i \) need not be bounded. Hence we can not always choose a subsequence of \( \{ S_i \} \) which converges analytically. Such an example was given in former paper ([6]).

**References**