60. Structure Theorems for Some Classes of Operators

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1. We consider bounded linear operators on a Hilbert space $H$. Denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_c(T)$ the spectrum, the point spectrum, the residual spectrum and the continuous spectrum respectively, by $r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}$ the spectral radius and by $W(T) = \{ (Tx, x) : \|x\| = 1 \}$ the numerical range. It is known [3] that $W(T)$ is convex and $\sigma(T) \subseteq \overline{W(T)}$ ($\text{conv} = \text{convex hull}, \overline{\text{cl}} = \text{closure}$). An operator $T$ is said to be hyponormal if $T^* T - T T^* \geq 0$, or equivalently if $\|T^* x\| \leq \|T x\|$ for every $x \in H$. As in [1] an operator is said to be restriction-convexoid (reduction-convexoid) if the restriction of $T$ to every invariant (invariant under $T$ and $T^*$) subspace is convexoid, where convexoid means that $\sigma(T) = \overline{W(T)}$.

In this Note we give some theorems on structure of hyponormal and restriction-convexoid operators whose spectrum lies on a convex curve.

2. Our main result in this section is

Theorem 1. If $T$ is a hyponormal operator and has the following properties

1° $T^p = S T^* S^{-1} + C$ for some $S$ for which $0 \in \overline{W(S)}$ and $C$ is compact operator

2° if $\mu, \lambda \in \sigma(T), 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots + \left(\frac{\lambda}{\mu}\right)^{p-1} \neq 0$

then $T$ is a normal operator.

For the proof we need the following

Lemma 1. If $T$ is a hyponormal operator which is the sum of a self-adjoint operator $A$ and a compact operator $C$, then $T$ is a normal operator.

Proof. Since $T$ is hyponormal it is known [10] that $T$ can be expressed uniquely as a direct sum $T = T_1 \oplus T_2$ defined on a product space $H = H_1 \oplus H_2$ where $H_i$ is spanned by all the proper vectors of $T$ such that: (a) $T_1$ is normal and $\sigma(T_1) = \overline{W(T)}$, (b) $T_2$ is hyponormal and $\sigma_p(T_2) = \varnothing$, (c) $T$ is normal if and only if $T_2$ is normal.

From the fact that $T = A + C$ we conclude by Lemma 2 [10] that $\sigma(T_2) \subseteq \sigma(A)$ and therefore $\sigma(T_2)$ is real. Since $\sigma_c(T)$ is open [9] and (T) is closed, we have that $\partial_s(T_2) \subseteq \sigma_p(T_2) \cup \sigma_c(T_2) = \sigma_c(T_2) (\partial = \text{boundary})$. Therefore $T_2$ is selfadjoint since $T_2$ is hyponormal with real spectrum.
Proof of the theorem. If $I(H)$ is the ideal of compact operators and $\omega(T)$, the Weyl spectrum (this means $\omega(T) = \bigcap_{C \in I(H)} \sigma(T + C)$) we obtain by the same reason as in [4] that $\omega(T)$ is real. By a result of Coburn [2] we conclude that $\sigma(T) = \omega(T) \cup \sigma_{p}(T)$ where $\sigma_{p}(T)$ contains only isolated eigenvalues of finite multiplicity. Let $T$, the restriction of $T$ to the space $H_{1}$ generated by eigenvectors corresponding to eigenvalues $\lambda \in \sigma_{p}(T)$. If we denote $H_{1} = H_{1}^{\perp}$, we obtain

$$H = H_{1} + H_{2}$$

and if $C = T_{1} \oplus 0$ and $A = 0 \oplus T_{2}$ we obtain $T = A + C$ where $A$ is self-adjoint and $C$ is compact (with finite range) and by Lemma 1 it follows that $T$ is a normal operator.

Corollary. If $T$ is a hyponormal operator with compact imaginary part, then $T$ is normal.

It is easy to see that for every operator we have

$$T = T^{*} + 2i \text{ Im } T$$

and by Theorem 1 for $p = 1$ the corollary follows.

Another proof of this corollary is in [7] and [10]

3. Theorem 2. If a reduction-convexoid operator $T$ whose spectrum lies on a convex curve is the sum of a compact operator $C$ and a generalized nilpotent operator $Q$ then $T$ is normal.

Proof. Since $T$ is convexoid and $\sigma(T)$ lies on a convex curve, $T$ can be expressed as a direct sum $T_{1} \oplus T_{2}$ defined on a product space $H_{1} \oplus H_{2}$, where $H_{1}$ is spanned by all the eigenvectors of $T$, such that $T_{1}$ is normal with $\sigma(T_{1}) = \sigma_{r}(T)$. By Weyl’s Theorem [3 problem 143] we conclude that $\sigma(T) \subset \sigma(Q) = \{0\}$ except the eigenvalues which implies $\sigma(T_{2}) \subset \sigma_{r}(T)_{2}$ since $\sigma_{r}(T_{2}) = \sigma_{r}(T_{2}) \subset \sigma_{r}(T)$. By Lemma 6[6], $H_{1}$ reduces $T$ and thus $T_{2}$ is convexoid operator with a single point in the spectrum. Since this point is zero we conclude that $T_{2} = 0$ which implies $H_{2} = 0$ and $T$ is normal.

We recall that an operator $T$, $\|T\| \leq 1$ and $\sigma(T) \subset \{z : |z| = 1\}$ is called unimodular contraction.

Corollary 1. If $T$ is a convexoid unimodular contraction and $T = C + Q$ then $T$ is unitary.

Corollary 2. If a reduction-convexoid operator $T$ with compact imaginary part has the spectrum on a convex curve, then $T$ is normal.

Proof. By Weyl’s Theorem $\sigma(T)$ is real except the eigenvalues and we conclude as above that $\sigma(T_{2})$ is real and convexoid. Therefore $T_{1}$ is selfadjoint and by Theorem 2 [6] $T$ is normal.

Theorem 3. If $T$ has the following properties:

1° spectral (in the sense of Dunford)

2° is restriction-convexoid with compact imaginary part, then there exists a direct decomposition of $H$, $H = H_{1} + H_{2} + \cdots$ such
that
a) $H_i, i=1, 2, \ldots$ is invariant under $T$

b) $T|_{H_i}$ is scalar ($T|_{H_i}$ is the restriction of $T$ to $H_i$)

c) $T|_{H_i} = \mu_i I, i=1, 2, \ldots, \mu_i$ complex numbers.

Proof. Since $T$ is almost normal [8], it follows that there exists a direct decomposition of $H$ with properties a) and b) and

$$T|_{H_i} = \mu_i I + Q_i,$$

$Q_i$ are compact nilpotent operators.

But $T|_{H_i}$ is convexoid and therefore $T|_{H_i} - \mu_i I$ is also convexoid with a single point (zero) in the spectrum. Then

$$T|_{H_i} - \mu_i I = 0$$

**Theorem 4.** If a restriction-convexoid operator whose spectrum lies on a convex curve is polynomially compact, then $T$ is normal.

Proof. By Theorem 2 [6] we have that $T = T_1 + T_2$ as above with the same properties. Since $H_i$ is invariant under $T$ and $T$ is polynomially compact then $\sigma(T_i) \subseteq \{ \lambda : p(\lambda) = 0 \}$ where $p(.)$ is a polynomial with $p(T) =$ compact. Therefore $\sigma(T_i)$ is a finite set and thus $H_i = \{0\}$. Indeed, in the contrary case, since $T$ is restriction convexoid we have that $\sigma(T_1) = \sigma_0(T_2)$ which is a contradiction.

References