127. Surjectivity of Linear Mappings and Relations

By Shouro Kasahara
Kobe University

In [3], Pták has proved the following theorem, in which (1) is called the closed relation theorem and (2) the open mapping theorem.

Theorem A. Let \( E \) be a Banach space, \( F \) a normed linear space, \( R \) a closed linear subspace of \( E \times F \), \( T \) a continuous linear mapping of \( E \) into \( F \), and let \( 0 < \alpha < \beta \). Let \( U \) and \( V \) be the unit balls of \( E \) and \( F \) respectively.

1. If the set \( RU + \alpha V \) contains a translate of \( \beta V \), then \( RE = F \) and \( (\beta - \alpha)V \subseteq RU \).

2. If the set \( T(U) + \alpha V \) contains a translate of \( \beta V \), then \( T(E) = F \) and \( (\beta - \alpha)V \subseteq T(U) \), so that \( T \) is open.

A theorem which is similar to the assertion (2) is obtained by McCord [2]:

Theorem B. Suppose \( T \) is a continuous linear mapping of a Banach space \( E \) into a normed linear space \( F \), for which there are positive real numbers \( \alpha \) and \( \beta \), \( \beta < 1 \), such that the following holds. For each \( y \) in \( F \) of norm 1, there exists an \( x \) in \( E \) of norm \( < \alpha \) such that \( y - Tx < \beta \). Then for each \( y \) in \( F \), there exists an \( x \) in \( E \) such that \( y = Tx \) and \( \|x\| < \alpha(1 - \beta)^{-1}\|y\| \).

Theorem A has been generalized by Baker [1]. In this paper we shall state other generalizations of Theorem A and a generalization of Theorem B.

Throughout this paper, vector spaces are over the real or the complex numbers. Let \( E \) and \( F \) be two vector spaces, \( A \) a subset of \( E \), and \( R \) be a subset of \( E \times F \). By \( R(A) \) we denote the set of all \( y \in F \) such that \( (x, y) \in R \) for some \( x \in A \); the set \( R([x]) \), where \( x \in E \), will be denoted by \( R(x) \). \( S(A) \) denotes the union of all \( \lambda A \) with \( \lambda \) in the closed unit interval \([0, 1]\), and \( A \) is said to be star-shaped if \( S(A) = A \).

The essential part of our results is included in the following

Lemma. Let \( E \) and \( F \) be two topological vector spaces, and \( R \) be a closed vector subspace of \( E \times F \). Let \( B_\alpha \) be a sequentially complete bounded star-shaped convex subset of \( E \) such that \( R(B_\alpha) \neq \emptyset \), and let \( B \) be a bounded subset of \( F \). Then \( B \subseteq R(B_\alpha) + \alpha B \) implies \( (1 - \alpha)B \subseteq R(B_\alpha) \) for every \( \alpha \in [0, 1] = [0, 1]\setminus\{1\} \).

Proof. It suffices to consider the case where \( \alpha \neq 0 \). Let \( y \) be an arbitrary element of \( B \). Since \( B \subseteq R(B_\alpha) + \alpha B \), there are points \( x_1 \in B_\alpha \)
and \( y_1 \in R(x_1) \) such that \( y - y_1 \in \alpha B \subseteq R(\alpha B_0) + \alpha^2 B \). Therefore we have, for some \( x_2 \in \alpha B_0 \) and for some \( y_2 \in R(x_2), \ y - y_1 - y_2 \in \alpha^2 B \subseteq R(\alpha^2 B_0) + \alpha^3 B \). Thus we can find recursively two sequences \( \{x_n \mid n=1, 2, \ldots \} \) and \( \{y_n \mid n=1, 2, \ldots \} \) satisfying the following conditions 1)–3) for every \( n=1, 2, \ldots \)

1) \( x_n \in \alpha^{n-1} B_0 \).

2) \( y_n \in R(x_n) \).

3) \( y - \sum_{i=1}^n y_i \in \alpha^n B \).

Since \( B_0 \) is star-shaped and convex, we have

\[
\sum_{i=1}^n x_i \in B_0 + \alpha B_0 + \cdots + \alpha^{n-1} B_0 \subseteq \frac{1-\alpha^n}{1-\alpha} B_0 \subseteq \frac{1}{1-\alpha} B_0
\]

for every \( n=1, 2, \ldots \). From the boundedness of \( B_0 \) and 1), it is easy to see that the sequence \( \{\sum_{i=1}^n x_i \mid n=1, 2, \ldots \} \) is a Cauchy sequence. Consequently, the sequence \( \{\sum_{i=1}^n x_i \mid n=1, 2, \ldots \} \) converges to an element \( x \in \frac{1}{1-\alpha} B_0 \), since \( B_0 \) is sequentially complete. Now the condition 2) shows \( \{\sum_{i=1}^n x_i, \sum_{i=1}^n y_i \} \in R \), and the condition 3) implies, because of the boundedness of \( B \), that the sequence \( \{\sum_{i=1}^n y_i \mid n=1, 2, \ldots \} \) converges to \( y \).

Thus we have \((x, y) \in R \) or \((1-\alpha)y \in R(B_0) \), which establishes the lemma.

If \( y \in R(x) \), then we have \( B \subseteq R(B_0) + \alpha B \) for \( B_0 = S(x), \ B = \{y\} \), and \( \alpha = 0 \). Therefore we have the following

**Theorem 1.** Let \( E \) and \( F \) be two topological vector spaces, \( R \) a closed vector subspace of \( E \times F \), and let \( y \in F \). If \( y \in R(x) \) for some \( x \in E \), then

\( (*) \) there exist a sequentially complete bounded star-shaped convex subset \( B_0 \) of \( E \) and a bounded subset \( B \subseteq F \) containing \( y \) such that \( R(B_0) \neq \emptyset \) and \( B \subseteq R(B_0) + \alpha B \) for some \( \alpha \in [0, 1) \).

Conversely, if the condition \( (*) \) is satisfied, then \( y \in R(x) \) for some \( x \in \frac{1}{1-\alpha} B_0 \).

The graph of a continuous linear mapping of a topological vector space \( E \) into a Hausdorff topological vector space \( F \) is a closed vector subspace of \( E \times F \). Hence the following corollary is evident.

**Corollary 1.** Let \( u \) be a continuous linear mapping of a topological vector space \( E \) into a Hausdorff topological vector space \( F \), and let \( y \in F \). If \( y = u(x) \) for some \( x \in E \), then

\( (**) \) there exist a sequentially complete bounded star-shaped con-
vex subset $B_0$ of $E$ and a bounded subset $B \subset F$ containing $y$ such that $B \subset u(B_0) + \alpha B$ for some $\alpha \in [0, 1)$.

Conversely, if the condition $(**)$ is satisfied, then $y = u(x)$ for some $x \in \frac{1}{1-\alpha}B_0$.

Theorem 1 yields obviously the following

**Theorem 2.** Under the hypothesis of Theorem 1, $R(E) = F$ if and only if there exists a family $\mathcal{B}$ of bounded subsets of $F$ such that the union of all members of $\mathcal{B}$ spans $F$ and there correspond, to each $B \in \mathcal{B}$, a sequentially complete bounded star-shaped convex subset $B_0$ of $E$ and an $\alpha \in [0, 1)$ satisfying the relations: $R(B_0) \neq \emptyset$ and $B \subset R(B_0) + \alpha B$.

**Corollary 2.** Under the hypothesis of Corollary 1, the mapping $u$ is surjective if and only if there exists a family $\mathcal{B}$ of bounded subsets of $F$ such that the union of all members of $\mathcal{B}$ spans $F$ and there correspond, to each $B \in \mathcal{B}$, a sequentially complete bounded star-shaped convex subset $B_0$ of $E$ and an $\alpha \in [0, 1)$ satisfying the relation $B \subset R(B_0) + \alpha B$.

Another consequence of Theorem 1 is the following

**Theorem 3.** Under the hypothesis of Theorem 1, if there exist a sequentially complete bounded star-shaped convex subset $B_0$ of $E$, a bounded subset $B$ of $F$, a subset $A$ of $F$ absorbing each non-zero element of $F$, and an $\alpha \in [0, 1)$ such that $R(B_0) \neq \emptyset$, $B \subset S(A)$ and $A \subset R(B_0) + \alpha B$, then $(1-\alpha)A \subset R(B_0)$, and hence $R(E) = F$.

In fact, since $R(B_0)$ is star-shaped convex, we have $S(B) \subset S(A) \subset R(B_0) + \alpha S(B)$, and so $(1-\alpha)S(B) \subset R(B_0)$; consequently we have

$$
(1-\alpha)A \subset (1-\alpha)S(A) \subset (1-\alpha)R(B_0) + \alpha(1-\alpha)S(B) \\
\subset (1-\alpha)R(B_0) + \alpha R(B_0) \subset R(B_0).
$$

The following corollary is a generalization of Theorem B.

**Corollary 3.** Under the hypothesis of Corollary 1, if there exist a sequentially complete bounded star-shaped convex subset $B_0$ of $E$, a bounded subset $B$ of $F$, a subset $A$ of $F$ absorbing each non-zero element of $F$, and an $\alpha \in [0, 1)$ such that $B \subset S(A)$ and $A \subset u(B_0) + \alpha B$, then $(1-\alpha)A \subset u(B_0)$, and so $u$ is surjective.

**Remark.** The well-known fact “a Hausdorff topological vector space $E$ having a precompact neighborhood of 0 is of finite dimensional” follows immediately from the above lemma. In fact, since a precompact neighborhood is bounded, it is sufficient to show that if a bounded set $B$ of $E$ is covered by a finite number of translations of $\alpha B$ for some $\alpha \in [0, 1)$, then $B$ spans a finite dimensional vector subspace of $E$. Now let $B \subset \bigcup_{i=1}^{n}(a_i + \alpha B)$, $a_1, a_2, \ldots, a_n \in E$. Then $B \subset e(S(A)) + \alpha B$, where $A$ is the convex hull of the set $\{a_1, \ldots, a_n\}$ and $e$ is the
identity mapping of $E$ into itself. Since $S(A)$ is sequentially complete and bounded, by the above lemma we have $(1-\alpha)B \subseteq S(A)$ from which the desired conclusion follows.

References