128. On Some Properties of $A^p(G)$-algebras

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1. Introduction. Let $G$ be a locally compact abelian group with dual group $\hat{G}$. We denote $dx$ and $d\hat{x}$ the Haar measures of $G$ and $\hat{G}$ respectively. Recently, Larsen, Liu, and Wang [4] have investigated a space $A^p(G)$ consisting of all complex-valued functions $f \in L^1(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$ $(p > 1)$. In this paper, we shall show further investigations of the algebra $A^p(G)$ proving the existence of the approximate identities of $A^p(G)$ and using the approximate identity to give a reproof of Theorem 5 in [4]. We show also that the closed primary ideal of $A^p(G)$ is maximal.

2. The approximate identities of $A^p(G)$-algebras. It is clear that $A^p(G)$ is an ideal dense in $L^1(G)$ under convolution. Indeed, for any $f \in A^p(G)$ and $g \in L^1(G)$,

$$\|f * g\|_p \lesssim \|g\|_{\infty} \|\hat{f}\|_p,$$

proving $f * g \in A^p(G)$ and the density of $A^p(G)$ in $L^1(G)$ follows from the fact that if $\{e_a\}$ is an approximate identity in $L^p(G)$ whose Fourier transforms have compact supports then $e_a \in A^p(G)$ and for an arbitrary function $f \in L^1(G)$ we have

$$f * e_a \in A^p(G) \text{ and } \|f * e_a - f\|_p \to 0.$$

Define the norm of $f \in A^p(G)$ $(1 \leq p < \infty)$ by

$$\|f\|_p = \|f\|_1 + \|\hat{f}\|_p,$$

where $\|f\|_1 = \int_G |f(x)| \, dx$ and $\|\hat{f}\|_p = \left( \int_{\hat{G}} |\hat{f}(\hat{x})|^p \, d\hat{x} \right)^{1/p}$. Then $A^p(G)$ is a commutative Banach algebra under convolution as its product and with the norm $\cdot \|_p$ (see [4; Theorem 3]).

We say here an approximate identity for $A^p(G)$ a family $\{e_a\}$ of functions in $A^p(G)$ such that for any $f \in A^p(G)$ and $\varepsilon > 0$, there exists $e_a \in \{e_a\}$ such that $\|e_a * f - f\|_p < \varepsilon$.

Theorem 1. The Banach algebra $A^p(G)$ has an approximate identity with the properties that it is also the bounded approximate identity for $L^1(G)$ and whose Fourier transform has compact support in $\hat{G}$.

Proof. By Rudin [7] Theorem 2.6.6, we see that there is a

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bounded approximate identity \( \{e_a\} \) in \( L^1(G) \) such that each \( e_a \) has compact support in \( \hat{G} \) where \( \{\alpha\} \) is a directed set. Let \( K \) be an arbitrary compact set in \( \hat{G} \). Then there exists a function \( k \in L^1(G) \) such that \( \hat{k}=1 \) on \( K \). Thus

\[
e_{\alpha} * k(\hat{x}) = \hat{e}_a(\hat{x}) \hat{k}(\hat{x}) = \hat{e}_a(\hat{x})
\]

for any \( \hat{x} \in K \). Now for a given \( \varepsilon > 0 \) there exists an index \( \alpha_0 \) such that \( \| e_a * k - k \|_1 < \varepsilon \) whenever \( \alpha > \alpha_0 \). Then for any \( \hat{x} \in K \),

\[
| \hat{e}_a(\hat{x}) - 1 | = | \hat{e}_a(\hat{x}) \hat{k}(\hat{x}) - \hat{k}(\hat{x}) | \leq \| \hat{e}_a \hat{k} - \hat{k} \|_\infty
\]

for \( \alpha > \alpha_0 \). Hence \( \hat{e}_a \) converges to 1 uniformly on any compact set in \( \hat{G} \). We assert that \( \{e_a\} \) becomes an approximate identity for \( A^p(G) \) as follows.

Since \( f \in A^p(G) \), \( \hat{f} \in L^p(\hat{G}) \). Therefore for a given \( \varepsilon > 0 \), we may choose a compact set \( K=K_\alpha \) in \( \hat{G} \) so that

\[
\int_{-K} |\hat{f}(\hat{x})|^p d\hat{x} < \frac{\varepsilon}{2^{p+1} M^p}
\]

where \( -K \) is the complement of the set \( K \) and \( M \) is a constant such that \( \| e_a \|_1 \leq M \). As \( \hat{e}_a \to 1 \) uniformly on \( K \),

\[
\int_K |\hat{f}(\hat{x}) \hat{e}_a(\hat{x}) - \hat{f}(\hat{x})|^p d\hat{x} \to 0.
\]

Thus there exists \( \alpha_0 \) such that

\[
\int_K |\hat{f}(\hat{x}) \hat{e}_a(\hat{x}) - \hat{f}(\hat{x})|^p d\hat{x} < \frac{\varepsilon}{2^{p+1}}
\]

for \( \alpha > \alpha_0 \) and so

\[
\int_{\hat{G}} |\hat{f}(\hat{x}) \hat{e}_a(\hat{x}) - \hat{f}(\hat{x})|^p d\hat{x} = \int_K |\hat{f}(\hat{x}) \hat{e}_a(\hat{x}) - \hat{f}(\hat{x})|^p d\hat{x} + \int_{-K} |\hat{f}(\hat{x}) \hat{e}_a(\hat{x}) - \hat{f}(\hat{x})|^p d\hat{x}
\]

\[
< \int_K |\hat{f}(\hat{x}) \hat{e}_a(\hat{x}) - \hat{f}(\hat{x})|^p d\hat{x} + 2p M^p \int_{-K} |\hat{f}(\hat{x})|^p d\hat{x}
\]

\[
< \frac{\varepsilon}{2^{p+1}} + \frac{\varepsilon}{2^{p+1}} = \frac{\varepsilon}{2^p}
\]

whenever \( \alpha > \alpha_0 \). Therefore

\[
\| f \hat{e}_a - \hat{f} \|_p < \frac{\varepsilon}{2} \quad \text{for} \quad \alpha > \alpha_0.
\]

On the other hand, since \( \{e_a\} \) is an approximate identity for \( L^1(G) \), there is an index \( \alpha_1 \) such that

\[
\| f \hat{e}_a - \hat{f} \|_1 < \frac{\varepsilon}{2} \quad \text{for} \quad \alpha > \alpha_1.
\]

Letting \( \alpha_2 = \sup (\alpha_0, \alpha_1) \), we obtain

\[
\| f \hat{e}_a - \hat{f} \|_p = \| f \hat{e}_a - f \|_1 + \| \hat{f} \hat{e}_a - \hat{f} \|_p
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

whenever \( \alpha > \alpha_2 \). This completes the proof. Q.E.D.

We notice that contrary to the usual case the approximate identity in \( A^p(G) \) can not be chosen to be uniformly bounded in general. Indeed,
let \( \{e_a\} \) be an arbitrary approximate identity for \( A^p(G) \), then \( \{e_a\} \) is also an approximate identity for \( L^p(G) \). Consider a compact set \( K \) with positive measure. As we have shown before, \( \hat{e}_a \to 1 \) uniformly on \( K \). Hence for any \( \varepsilon > 0 \) there exists \( \alpha \) such that

\[
\int_K (|\hat{e}_a(\hat{x})|^p - 1)d\hat{x} > -\varepsilon
\]
or

\[
\int_K |\hat{e}_a(\hat{x})|^p d\hat{x} > m(K) - \varepsilon
\]
where \( m(K) \) is the measure of \( K \). Therefore

\[
\|e_a\|_p > (m(K) - \varepsilon)^{1/p} > (m(K)/2)^{1/p}
\]
for a small \( \varepsilon \), and in general \( m(K) \) can be large enough. Hence \( \{\|e_a\|_p\} \) is not uniformly bounded so is not \( \{\|e_a\|^p\} \).

Applying Theorem 1, we can reprove the following result due to Larsen, Liu, and Wang [4; Theorem 5].

**Theorem 2.** For each \( p \) (\( 1 \leq p < \infty \)) the following two statements hold:

(i) \( \text{If } I_1 \text{ is a closed ideal in } L^1(G), \text{ then } I_1 = I_1 \cap A^p(G) \text{ is a closed ideal in } A^p(G). \)

(ii) \( \text{If } I_1 \text{ is a closed ideal in } A^p(G) \text{ and } I_1 \text{ is the closure of } I \text{ in } L^1(G), \text{ then } I_1 \text{ is a closed ideal in } L^1(G) \text{ and } I = I_1 \cap A^p(G). \)

**Remark.** This theorem is also suggested by analogous results in Liu and Wang [5, Theorem 7] in which \( A^p(G) \) is replaced by \( D = D_{1,p} = L^p(G) \cap L^p(G) \) (\( 1 < p < \infty \)) with the norm \( \|f\| = \max(\|f\|_1, \|f\|_p) \).

**Proof of Theorem 2.** The proof of (i) is immediate and will be omitted. Similarly, in (ii) it is easy to verify that \( I_1 \) is a closed ideal in \( L^1(G) \) and that \( I \subset I_1 \cap A^p(G). \) We shall prove \( I = I_1 \cap A^p(G) \) as following.

It suffices to prove that \( I \) is dense in \( I_1 \cap A^p(G). \) Let \( f \in I_1 \cap A^p(G). \) We show that for any \( \varepsilon > 0 \) there is an element \( h \) in \( I \) such that \( \|h - f\|^p < \varepsilon. \) By Theorem 1, there exists an approximate identity \( \{e_a\} \) of \( A^p(G) \) for which each \( \hat{e}_a \) has compact support in \( \hat{G}. \) Take a sequence \( \{f'_n\} \) in \( I \) such that \( f'_n \to f \) in \( L^1\)-norm. It follows that

(1) \( e_a * f'_n \to e_a * f \)

in \( L^1\)-norm for each fixed \( \alpha. \) Now, since

\[
|e_a * f'_n(\hat{x}) - e_a * f(\hat{x})|^p \leq |\hat{e}_a(\hat{x})|^p |\hat{f}'_n(\hat{x}) - \hat{f}(\hat{x})|^p
\leq |\hat{e}_a(\hat{x})|^p \|\hat{f}'_n - \hat{f}\|^p
\leq |\hat{e}_a(\hat{x})|^p \|f'_n - f\|^p
\leq M |\hat{e}_a(\hat{x})|^p
\]
for some constant \( M \) and \( \hat{e}_a \in L^p(\hat{G}), \) the Lebesgue convergence theorem is applicable, and

(2) \( \int_G |e_a * f'_n(\hat{x}) - e_a * f(\hat{x})|^p d\hat{x} \to 0 \quad \text{as } n \to \infty \)
for each $\alpha$. Given any $\varepsilon > 0$, there exists $\alpha_0$ such that
\[
\| e_{\alpha_0} * f - f \|_p < \varepsilon / 3
\]
and for this $e_{\alpha_0}$ there is an integer $n_0$ such that, by (1) and (2)
\[
\| e_{\alpha_0} \hat{f} |_{n_0} - \hat{e_{\alpha_0}} \hat{f} \|_p < \varepsilon / 3
\]
and
\[
\| e_{\alpha_0} * f \|_{n_0} - e_{\alpha_0} * f \|_1 < \varepsilon / 3.
\]
Therefore
\[
\| e_{\alpha_0} * f \|_{n_0} - f \|_1 \leq \| e_{\alpha_0} * f \|_{n_0} - e_{\alpha_0} * f \|_p + \| e_{\alpha_0} * f - f \|_p
\]
\[
\leq \| e_{\alpha_0} * f \|_{n_0} - e_{\alpha_0} * f \|_1 + \| \hat{e_{\alpha_0}} \hat{f} |_{n_0} - \hat{e_{\alpha_0}} \hat{f} \|_p + \| e_{\alpha_0} * f - f \|_p
\]
\[
< \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon.
\]
Since $e_{\alpha_0} * f |_{n_0} \in I$, this completes the proof. Q.E.D.

3. Primary ideals in $A^p(G)$. A primary ideal of a Banach algebra $B$ means a proper ideal $I$ in $B$ that is contained in only one maximal ideal of $B$.

In the Gel'fand representation $\hat{B}$, a primary ideal $I$ can be characterized by the fact that the set on which all functions $\hat{f}(M) \in I$ (where $M$ is a maximal ideal) vanish consists of a single point. If $I$ is a closed primary ideal, then the residue-class algebra $B/I$ contains a unique maximal ideal. The conclusion of Theorem 2 holds also for the case of primary ideals. As the (regular) maximal ideal space of $L^1(G)$ is homeomorphic to the (regular) maximal ideal space of $A^p(G)$, the following proposition holds immediately.

**Proposition 3.** There is a one-to-one correspondence between the set of all closed primary ideals of $A^p(G)$ and the set of all closed primary ideals of $L^1(G)$. More precisely, every closed primary ideal of $A^p(G)$ is simply the intersection of a unique closed primary ideal of $L^1(G)$ with $A^p(G)$.

Kaplansky [3] proved that every closed primary ideal in $L^1(G)$ is maximal; and so we have immediately that

**Proposition 4.** Every closed primary ideal in $A^p(G)$ is maximal; therefore we can identify the set of all closed primary ideals in $A^p(G)$ with $\hat{G}$.

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**References**


