243. On Quasi- k -Spaces

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0. Introduction. In this paper, we shall treat the case that the product space is a quasi- k -space. In Section 1, we give definitions and preliminaries. In Section 2, we shall prove the following theorems;

(a): A space X is locally countably compact if and only if $X \times Y$ is a quasi- k -space for every sequential space Y .

(b): Let X and Y be sequential spaces. Then $X \times Y$ is a sequential space if and only if it is a quasi- k -space.

(c): If X is a sequential q -space, and Y is a k -space and a q -space, then $X \times Y$ is a k -space and a q -space.

Finally, in Section 3, we consider the product space of uncountably many spaces.

We assume all spaces are regular and T_2 , and all maps are continuous and onto. The weak topology in the sense of J. Dugundji [5], will be used throughout this paper.

1. Definitions and preliminaries. A space X is called a quasi- k -space (sequential space) if a subset F of X is closed whenever $F \cap C$ is closed in C for every countably compact (compact metric) subset C of X by J. Nagata [12] (S. P. Franklin [6]) respectively. Quasi- k -spaces (Sequential spaces) are precisely the quotients of M -spaces defined by K. Morita [11] (metric spaces) respectively. Of course, sequential spaces are k -spaces and k -spaces are quasi- k -spaces. But the converses do not hold. Indeed, the Stone-Čech compactification of a normal and non-compact space is not sequential, and a countably compact space A_1 constructed by J. Novák [13] is not a k -space.

Lemma 1.1. *Let $f_i: X_i \rightarrow Y_i$ ($i=1, 2$) be quotient and X_1 be sequential. If $Y_1 \times Y_2$ has the weak topology with respect to $\mathcal{F} = \{Y_1 \times C; C \text{ is closed countably compact in } Y_2\}$, then $f_1 \times f_2$ is quotient. Especially, when f_2 is closed, the closedness of the subset C is omitted.*

Proof. From the fact that $f_2|_{f_2^{-1}(C)}$ is quotient for every closed subset C of Y_2 , and that if $f_1 \times f_2|(f_1 \times f_2)^{-1}(F)$ is quotient for every $F \in \mathcal{F}$, then $f_1 \times f_2$ is quotient, we can assume Y_2 is countably compact. Moreover, sequential spaces are the quotients of locally compact metric spaces by S. P. Franklin [6; Corollary 1.13], we can also assume X_1 is locally compact metric. Now, $f_1 \times f_2 = (f_1 \times i_{Y_2}) \cdot (i_{X_1} \times f_2)$ and $i_{X_1} \times f_2$ is quotient by J. H. C. Whitehead [15; Lemma 4], and $f_1 \times i_{Y_2}$ is quotient by E. Michael [9; Theorem 4.1], $f_1 \times f_2$ is quotient.

Lemma 1.2. (a) *If X is a sequential space, then a countably compact subset of X is always closed.* (b) *Let X be either normal or countably paracompact. If C is a countably compact subset of X , then $\bar{C}$ is also countably compact.*

Proof. (a) is easily proved by the definition of a sequential space. (b) Let X be normal. Assume that $\bar{C}$ is not countably compact. Then there exists a discrete set $\{x_i; i \in n\}$ contained in $\bar{C}$. Hence we can choose a discrete collection $\{V_i; i \in N\}$ of open subsets with $x_i \in V_i$ by the normality of X . Since $x_i \in \bar{C}$, there exists a sequence $\{y_i; i \in N\}$ with $y_i \in V_i \cap C$. But C is countably compact, the sequence $\{y_i; i \in N\}$ has a cluster point. This is impossible. In case X is a countably paracompact space, from F. Ishikawa [7], (b) is proved straightforwardly.

Remark. In completely regular spaces, (b) need not be true. Indeed, let $X = [0, \Omega] \times [0, \omega] - (\Omega, \omega)$, where ω is the first non-finite ordinal and Ω is the first uncountable ordinal, and let $A = [0, \Omega] \times [0, \omega]$. Then A is countably compact but $\bar{A} = X$ is not countably compact.

According to E. Michael [8], a space X is called a q -space if each point of X has a sequence $\{U_i; i \in N\}$ of open nbds (=neighborhoods) such that $\bar{U}_{i+1} \subset U_i$, and if $x_i \in U_i$, then the sequence $\{x_i; i \in N\}$ has a cluster point. Such a sequence is called a q -sequence of nbds. Locally countably compact spaces, M -spaces, and spaces of pointwise countable type defined by A. Arhangel'skii [1] are all q -spaces. A q -space is a quasi- k -space by J. Nagata [12].

Lemma 1.3. *if (a) $X \times Y$ is a q -space, or (b) $X \times Y$ is a quasi- k -space and Y is normal or countably paracompact, or sequential, then $X \times Y$ has the weak topology with respect to $\mathcal{C} = \{X \times C; C \text{ is closed countably compact in } X\}$.*

Proof. (a): Put $X_1 = X$ and $X_2 = Y$. Let F be a subset of $X_1 \times X_2$ such that $F \cap C$ is closed for every $C \in \mathcal{C}$. Let $(x_1, x_2) \in \bar{F}$ be given and V_i be any open nbd of x_i . Since $X_1 \times X_2$ is a q -space, it has a q -sequence $\{W_{1j} \times W_{2j}; j \in N\}$ of a point (x_1, x_2) . Let W'_{ij} be an open nbd of x_i such that $\bar{W}'_{ij+1} \subset W'_{ij} \subset V_i \cap W_{ij}$ for each $j \in N$. Since $(x_1, x_2) \in \bar{F}$, there exists a sequence $\{(x_{1j}, x_{2j}); j \in N\}$ with $(x_{1j}, x_{2j}) \in F \cap (W'_{1j} \times W'_{2j})$. Put $C_2 = \bigcap_{j=1}^{\infty} W_{2j}$ and $C'_2 = \{x_{2j}; j \in N\} \cup \bigcap_{j=1}^{\infty} W'_{2j}$, then C_2 and C'_2 are closed countably compact in X_2 , and the sequence $\{x_{1j}, x_{2j}; j \in N\}$, whose closure is contained in a closed subset $F \cap (X_1 \times C'_2)$, has a cluster point in $(V_1 \times V_2) \cap (X_1 \times C_2)$. Hence $(x_1, x_2) \in F$ by the closedness of $F \cap (X_1 \times C_2)$. Therefore $X_1 \times X_2$ has the weak topology with respect to $\mathcal{C}$. (b): Since every subset K of $X \times Y$ is contained in $X \times \overline{P_Y(K)}$, where P_Y is the projection of $X \times Y$ onto Y , and $P_Y(K)$ is closed countably compact for every countably compact subset K of $X \times Y$ by Lemma 1.2,

$X \times Y$ has the weak topology with respect to $\mathcal{C}$.

From T. Chiba [4; Theorem 4], Lemma 1.1, and Lemma 1.3, we have

Proposition 1.4. *Let $f_i: X_i \rightarrow Y_i$ ($i=1, 2$) be quotient maps and X_1 be a sequential space. If (a) Y_1 and Y_2 are q -spaces, or (b) $Y_1 \times Y_2$ is a quasi- k -space and Y_2 is either normal or countably paracompact, then $f_1 \times f_2$ is quotient.*

Remark. Lemma 1.1 remains true if we replace the words “sequential” and “closed countably compact” by “a k -space” and “compact” respectively, which leads to E. Michael [10; Theorem 1.5].

2. Proof of theorems.

Theorem 2.1. *The following properties of a space X are equivalent.*

- (a) X is a locally countably compact space.
- (b) $X \times Y$ is a quasi- k -space for every sequential space Y .
- (c) $X \times Y$ is a quasi- k -space for every paracompact sequential space Y .

Proof. (a) $\rightarrow$ (b): Let Y_0 be the topological sum of the family $\{C; C \text{ is compact metric in } Y\}$, and f be the quotient map of Y_0 onto Y . Then $f \times i_X$ is quotient by E. Michael [9; Theorem 4.1 (a) $\rightarrow$ (b)]. Since $Y_0 \times X$ is a quasi- k -space and quotients of quasi- k -spaces are quasi- k -spaces, $X \times Y$ is a quasi- k -space.

(b) $\rightarrow$ (c): Obvious.

(c) $\rightarrow$ (a): Assume that X is not countably compact.

From E. Michael [9; Theorem 4.1 (c) $\rightarrow$ (a)], there exists a closed map g of a metric space onto a space Y such that $i_X \times g$ is not quotient. Since Y is a paracompact sequential space, $X \times Y$ is a quasi- k -space by the hypothesis. From Lemma 1.1 and the proof of Lemma 1.3 (b), $i_X \times g$ is quotient, which is impossible.

Theorem 2.2. *Let X and Y be sequential spaces. Then $X \times Y$ is sequential if and only if it is a quasi- k -space.*

Proof. The “only if” part is obvious.

“if”: Let X_0 and Y_0 be the topological sums of the families $\{C; C \text{ is compact metric in } X\}$ and $\{C'; C' \text{ is compact metric in } Y\}$ respectively, and let $f: X_0 \rightarrow X, g: Y_0 \rightarrow Y$ be quotient. Since $X \times Y$ is a quasi- k -space and Y is sequential, from Lemma 1.1 and Lemma 1.3 (b), $f \times g$ is quotient. Since $X_0 \times Y_0$ is metric, $X \times Y$ is sequential.

From T. Chiba [4; Theorem 4] and J. Nagata [12; Corollary to Theorem 1], and Theorem 2.2, we have

Corollary 2.3. *If X and Y are sequential q -spaces, then $X \times Y$ is a sequential q -space.*

From Theorem 2.1 and Theorem 2.2, we have

Corollary 2.4 (T. K. Boehme [3]). *If X is a locally countably compact, sequential space and Y is a sequential space, then $X \times Y$ is a sequential space.*

According to P. Bacon [2], a space X is called isocompact if every closed countably compact subset of X is compact. Paracompact spaces, σ -spaces, and developable spaces are all isocompact spaces.

Theorem 2.5. *Let X be either normal or countably paracompact. If (a) X is isocompact and Y is a k -space, or (b) X is a k -space and Y is sequential, then $X \times Y$ is a k -space if and only if it is a quasi- k -space.*

Proof. Let $X \times Y$ be a quasi- k -spaces. (a) $X \times Y$ has the weak topology with respect to $\{\overline{P_x(K)} \times Y; K \text{ is countably compact in } X \times Y\}$, where P_x is the projection of $X \times Y$ onto X , and $\overline{P_x(K)}$ is compact for every countably compact subset K of $X \times Y$ by Lemma 1.2 (b). From Remark to Proposition 1.4, $X \times Y$ is a k -space by the same way as in the proof of Theorem 2.2. (b) Similarly, from Proposition 1.4 (b), $X \times Y$ is a k -space.

Remark. Lemma 1.3, Proposition 1.4, and Theorem 2.5 remain true if we replace the words "normal" and "countably paracompact" by "locally normal" and "locally countably paracompact" respectively.

According to A. Arhangel'skii [1], a space X is called of pointwise countable type if each point of X is contained in a compact subset having a countable local basis.

Theorem 2.6. *Let X be a k -space and a q -space. If Y is either a space of pointwise countable type or a sequential q -space, then $X \times Y$ is a k -space and a q -space.*

Proof. From T. Chiba [4; Theorem 4], $X \times Y$ is a q -space. If Y is of pointwise countable type, put $X = X_1$ and $Y = X_2$, then the subsets C_2, C'_2 in the proof of Lemma 1.3 (a) are compact. Hence $X \times Y$ has the weak topology with respect to $\{X \times C; C \text{ is compact in } Y\}$. Therefore, by the same way as in the proof of Theorem 2.5 (a), $X \times Y$ is a k -space. Similarly, if Y is a sequential q -space, $X \times Y$ is a k -space by Proposition 1.4 (a).

3. In this section, we consider the product space of uncountably many spaces.

From the fact that every subset K of $\prod_{\alpha \in A} X_\alpha$ is contained in $\prod_{\alpha \in A} P_\alpha(K)$, where P_α is the projection of $\prod_{\alpha \in A} X_\alpha$ onto X_α , and Lemma 1.2, we have

Theorem 3.1. *Let X_α be an isocompact space which is normal, or countably paracompact, or sequential for each $\alpha \in A$. Then $\prod_{\alpha \in A} X_\alpha$ is isocompact, and it is a k -space if and only if it is a quasi- k -space.*

Lemma 3.2. *If $\prod_{\alpha \in A} X_\alpha$ is a q -space, then all but a countable number of spaces X_α must be countably compact.*

Proof. Assume that there exists an uncountable subset A' of A such that $X_{\alpha'}$ is not countably compact for each $\alpha' \in A'$. Then $X_{\alpha'}$ contains a copy of N , say $N_{\alpha'}$, as a closed subset for each $\alpha' \in A'$. Since $\prod_{\alpha' \in A'} N_{\alpha'}$ is closed in $\prod_{\alpha \in A} X_{\alpha}$, it is a q -space. Then there exists a q -sequence $\mathcal{U} = \{U_i; i \in N\}$ of a point of $\prod_{\alpha' \in A'} N_{\alpha'}$, such that each U_i is an open basic subset of $\prod_{\alpha' \in A'} N_{\alpha'}$. Put $K = \bigcap_{i=1}^{\infty} U_i$, K is countably compact and for any open subset 0 of $\prod_{\alpha' \in A'} N_{\alpha'}$ containing K we can find U_i satisfying $K \subset U_i \subset 0$. Let $B = \{\alpha'; \alpha' \in A', P_{\alpha'}(U) \neq N_{\alpha'}, \text{ for some } U \text{ in } \mathcal{U}\}$, where $P_{\alpha'}$ is the projection of $\prod_{\alpha' \in A'} N_{\alpha'}$ onto $N_{\alpha'}$. Then there exists an element α'_0 in $A' - B$. Since $P_{\alpha'_0}(K)$ is compact, $P_{\alpha'_0}(K) \neq N_{\alpha'_0}$. Put $0 = P_{\alpha'_0}(K) \times \prod_{\alpha' \neq \alpha'_0} N_{\alpha'}$, 0 is an open subset of $\prod_{\alpha \in A} N_{\alpha}$ containing K . But $U_i \not\subset 0$ for each $i \in N$, which is impossible.

Theorem 3.3. *Let X_{α} be an isocompact space for each $\alpha \in A$. Then $\prod_{\alpha \in A} X_{\alpha}$ is a q -space if and only if it is of pointwise countable type. Especially, if each X_{α} is a paracompact M -space, then the following properties of the product space $\prod_{\alpha \in A} X_{\alpha}$ are equivalent.*

(a) a q -space., (b) an M -space., (c) a paracompact space.

Proof. Since spaces of pointwise countable type are q -spaces, the "if" part is proved. The "only if" part follows from the fact that isocompact q -spaces are of pointwise countable type, and A. Arhangel'skii [1; Theorem 3.9], and Lemma 3.2.

Since M -spaces are q -spaces, from K. Morita [11; Theorem 6.4] and Lemma 3.2, (a) $\leftrightarrow$ (b) and (b) $\rightarrow$ (c) are proved. (c) $\rightarrow$ (b) follows from A. H. Stone [14; Corollary to Theorem 4] and K. Morita [11; Theorem 6.4].

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