251. Some Existence Theorems in Cluster Set Theory

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1. Let C be the unit circle and D be the open unit disk in the complex plane.

Theorem 1. *There exists a holomorphic function f in D for which the set $I(f)$ of Plessner points [3, p. 147] is residual [3, p. 75] on C and of logarithmic measure [7, p. 64] zero.*

Theorem 2. *There exists a bounded univalent holomorphic function f in D for which the set $M(f)$ of Meier points [3, p. 153] is of logarithmic measure zero.*

Furthermore, we obtain some improvements of Bagemihl-Seidel's results [2, p. 191, Corollaries 3~5], one of which may be stated as

Theorem 3. *There exist a holomorphic function f in D and a subset S of C , being of logarithmic measure zero, such that the radial cluster set [3, p. 72] of f at any point of $C-S$ coincides with the unit circle.*

Remark 1. A bounded set of logarithmic measure zero is known to be of logarithmic capacity zero. In Remark 3 of the next section we ascertain this for our special example S .

I wish to express my warmest thanks to Prof. K. Hatano for valuable conversations.

2. We construct a subset S of C satisfying the following three conditions:

- (i) $C-S$ is of first Baire category on C .
- (ii) S is a G_δ subset of C .
- (iii) The logarithmic measure of S is zero.

Let $K=\{z_1, \dots, z_n, \dots\}$ be a countable dense subset of C and let $\varepsilon_1, \dots, \varepsilon_k, \dots$ be a sequence of positive numbers such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Let δ_{kn} be an open disk containing z_n whose radius is $r_{kn} = \exp(-2^n/\varepsilon_k)$ ($k, n=1, 2, \dots$). Let $\delta_k = \bigcup_{n=1}^{\infty} \delta_{kn}$ and let $\delta = \bigcap_{k=1}^{\infty} \delta_k$. Then $S = \delta \cap C$ is the desired one. Indeed, for any k , the closed set $C - \delta_k$ is nowhere dense on C since $\delta_k \cap C \supset K$ is open and dense on C . Therefore the set

$$(*) \quad C-S = \bigcup_{k=1}^{\infty} (C-\delta_k)$$

is of first category on C . To prove (iii) we use the same notation as in [7, p. 63 ff.] with $h(t) = \{\log(1/t)\}^{-1}$. We use "disks" instead of

“squares” in p. 63 of [7]; this is not an essential change. Then, since δ_{kn} ($n=1, 2, \dots$) cover S and $r_{kn} < \exp(-1/\varepsilon_k) \equiv \rho_k$, we have

$$H(\rho_k) \text{ (for } S) \leq \sum_{n=1}^{\infty} h(r_{kn}) = \sum_{n=1}^{\infty} (\varepsilon_k/2^n) = \varepsilon_k.$$

This means that $H(\rho_k) \rightarrow 0$ as $k \rightarrow \infty$, which, combined with $\rho_k \rightarrow 0$ as $k \rightarrow \infty$, proves that the logarithmic measure of S is zero.

Remark 2. Generalizations to higher dimensional spaces or to metric spaces under the condition that C is a perfect set in the considered spaces are fairly easy. Prof. Hatano communicated orally that the condition (iii) may well be generalized to “Hausdorff measure”.

Remark 3. By Erdős-Gillis’s theorem [7, p. 66] combined with (iii) the inner logarithmic capacity and hence the logarithmic capacity of S is zero since S is capacitable by (ii). Consequently our example is simpler and more general than Kishi-Nakai’s [4].

3. By Lappan’s theorem [5, Theorem 2] there exists a holomorphic function f in D for which $I(f)=S$ since S is a G_δ subset of C . This proves Theorem 1.

We apply Bagemihl’s theorem [1] to the set $C-S$. Then there exists a bounded univalent holomorphic function f in D such that $M(f) \cap (C-S) = \emptyset$ (empty). Theorem 2 follows from $M(f) \subset S$ and (iii).

Since $C-S$ is F_σ of first category and of the form of $(*)$ on C we have the improvements of Corollaries 3~5 in [2, p. 191] in the sense that the subsets of C of exceptional character are of logarithmic measure zero.

Remark 4. Bagemihl’s proof of his theorem in [1] depends in part on a theorem of Lohwater and Piranian [6, p. 7, Theorem 1’]. We remark that some theorems in [6] may well be applied to our F_σ set $C-S$. For example, we have:

There exists a bounded univalent holomorphic function f in D such that f has a radial limit $f(\zeta)$ at every point $\zeta \in C$ and that the function $f(\zeta)$ on C is discontinuous at every $\zeta \in C-S$ and is continuous at every $\zeta \in S$.

References

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