254. Ergodic Properties of Piecewise Linear Transformations

By Iekata SHIOKAWA

Department of Mathematics, Tokyo Metropolitan University

(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1970)

1. Introduction. After the work of Rényi [1], ergodic properties of β -expansions of real numbers have been studied in [2]–[4]. In this paper we generalize these results for a class of expansions, called piecewise linear expansions, which includes β -expansions as special cases.

Let $\bar{\beta} = (\beta_0, \beta_1, \dots, \beta_N)$, $N \geq 1$, be a $(N+1)$ -tuple of positive number such that $0 < \theta \equiv \beta_N(1 - \sum_{k=0}^{N-1} (1/\beta_k)) \leq 1$.

We denote the set of all $(N+1)$ -tuples by $V(N+1)$. For each $\bar{\beta} \in V(N+1)$, we define a corresponding function $f(t)$ by

$$f(t) = \begin{cases} \frac{t}{\beta_0}, & 0 \leq t \leq 1, \\ f(k) + \frac{t-k}{\beta_k}, & k < t \leq k+1, (k=1, 2, \dots, N+1), \\ 1, & N < t \leq N+\theta, (k=N), \\ 1, & t > N+\theta. \end{cases}$$

The function $f(t)$ satisfies the Rényi's conditions [1]. Thus every real number x has the f -expansion

$$x = a_0(x) + f(a_1(x) + f(a_2(x) + \dots)),$$

where the digits $a_n(x)$, $n=0, 1, \dots$, and the remainders

$$T^n x = f(a_n(x) + f(a_{n+1}(x) + \dots)), \quad n=0, 1, \dots,$$

are defined by the following recursive relations: $a_0(x) = [x]$, $T^0 x = \{x\}$, $T^{n+1} x = \{f^{-1}(T^n x)\}$, $a_{n+1}(x) = [f^{-1}(T^n x)]$, $n=0, 1, \dots$, where $[z]$ denotes the integral part and $\{z\}$ the fractional part of the real number z and f^{-1} is the inverse function of f .

This f -expansion is called a *piecewise linear expansion induced by $\bar{\beta}$* or simply *$\bar{\beta}$ -expansion*, and the transformation $Tx = \{f^{-1}(x)\}$, $0 \leq x < 1$, is called a *piecewise linear transformation induced by $\bar{\beta}$* . By definition, T is a many to one transformation of $[0, 1)$ onto itself and nonsingular with respect to the Lebesgue measure m .

For the number 1, we define, especially, $a_0(1) = 0$ and $T^0 1 = 1$. Then $\bar{\beta} \in V(N+1)$ is said to be *periodic* if the $\bar{\beta}$ -expansion of 1 has a recurrent tail, and *rational* if the $\bar{\beta}$ -expansion of 1 has a zero tail. The *order* of a rational $\bar{\beta}$ is the minimum integer r such that $a_n(1) = 0$ for all $n > r+1$.

2. Invariant measures. Lemma 1. *Let T be a piecewise linear transformation induced by $\bar{\beta} \in V(N+1)$ and μ a finite measure equivalent to the Lebesgue measure m . Then μ is T -invariant if and only if*

$$h(x) = \sum_{k=0}^N f'(k+x)h(f(k+x))dx, \text{ a.e.}$$

where $h(x)$ is the Radon-Nikodym derivative of μ .

Proof. For any $t \in [0, 1)$, we have

$$\mu(T^{-1}[0, 1)) = \int_0^t \sum_{k=0}^N f'(k+x)h(f(k+x))dx.$$

The lemma is an immediate conclusion of this fact.

For any $\bar{\beta} \in V(N+1)$, we define a function

$$h(x) = \sum_{n=0}^{\infty} \frac{C_n(x)}{\beta_{a_0(1)}\beta_{a_1(1)} \cdots \beta_{a_n(1)}},$$

where $\beta_{a_0(1)} = 1$ and $C_n(x)$ is the characteristic function of the interval $[0, T^n 1)$.

Theorem 1. Let T be a piecewise linear transformation induced by $\bar{\beta}$ and put $\mu(A) = \int_A h(x)dx$ for any measurable set A . Then μ is finite T -invariant measure equivalent to the Lebesgue measure.

Proof. First we prove that

$$(1) \quad \sum_{k=0}^N f'(k+x)C_n(f(k+x)) = f(a_{n+1}(1)) + \frac{C_{n+1}(x)}{\beta_{a_{n+1}(1)}}$$

If $f(x) > T^n 1$, then (1) is trivial. Thus it suffices to prove (1) when there exists an integer k such that $f(k+x) < T^n 1$. There are two possibilities: (i) there exists k such that $f(k+x) < T^n x < f(k+x)$, (ii) there exists k such that $f(k+1) \leq T^n 1 \leq f(k+1+x)$. In the case (i) $a_{n+1}(1) = k$, $C_{n+1}(x) = 1$, and in the case (ii) $a_{n+1}(1) = k+1$, $C_{n+1}(x) = 0$. As a result, we get (1). Furthermore, by the piecewise linearity of f , we have

$$(2) \quad 1 = \sum_{n=0}^{\infty} \frac{f(a_{n+1}(1))}{\beta_{a_0(1)}\beta_{a_1(1)} \cdots \beta_{a_n(1)}}.$$

Therefore, we have

$$\begin{aligned} & \sum_{k=0}^N f'(k+x)h(f(k+x)) \\ &= \sum_{n=0}^{\infty} \frac{1}{\beta_{a_0(1)}\beta_{a_1(1)} \cdots \beta_{a_n(1)}} \sum_{k=0}^N f'(k+x)C_n(f(k+x)) \\ &= \sum_{n=0}^{\infty} \frac{C_{n+1}(x)}{\beta_{a_0(1)}\beta_{a_1(1)} \cdots \beta_{a_{n+1}(1)}} + \sum_{n=0}^{\infty} \frac{f(a_{n+1}(1))}{\beta_{a_0(1)}\beta_{a_1(1)} \cdots \beta_{a_n(1)}} \quad (\text{by (1)}) \\ &= h(x) \quad (\text{by (2)}). \end{aligned}$$

this and Lemma 1 imply the theorem.

Corollary 1. $h(x)$ is a decreasing jump function which satisfies $1 = h(1) \leq h(x) \leq h(0) < \infty$, a.e.

Corollary 2. $h(x)$ is a step function with a finite number of steps if and only if $\bar{\beta}$ is periodic. Especially $h(x) = 1$ if and only if $\bar{\beta}$ is rational of order 0.

In what follows we shall investigate the transformation T with the normalized invariant measure $p(\cdot) = \mu(\cdot)/\mu([0, 1))$.

3. **Exactness.** A measure preserving transformation T on a Lebesgue space $(X, \mathcal{B}, P)$ is said to be exact if $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \{X, \emptyset\}$.

Rohlin's criterion [4]. Let $\mathcal{U}$ be a countable system of sets of positive measure on X such that the finite unions of pairwise disjoint sets $A \in \mathcal{U}$ form an ensemble everywhere dense in $\mathcal{B}$. If there exists a positive integer-valued function $n(A)$, $A \in \mathcal{U}$, and a positive number q such that $P(T^{n(A)}A) = 1$, $A \in \mathcal{U}$, and

$$(4) \quad P(T^{n(A)}E) \leq q \frac{P(E)}{P(A)},$$

for all measurable set $E \subset A$ with measurable image $T^{n(A)}E$, then T is exact.

Theorem 2. *Every piecewise linear transformation is exact.*

Proof. The proof is based on the Rohlin's criterion. Let $\bar{\beta} \in V(N+1)$, be given arbitrary and let us denote by ξ a partition of $(0, 1)$ into subintervals generated by the points $f(k)$, $k=1, 2, \dots, N$. We set $U_n = \{A \in T^{-(n-1)}\xi; TA \in T^{-(n-2)}\}$, $n=1, 2, \dots$, $U = \bigcup_{n=1}^{\infty} U_n$ and $n(A) = n$ if $A \in U_n$. Then, the density and the relation $P(T^{n(A)}A) = 1$, $A \in U$, are obviously satisfied. We must prove that there exists a constant $q = q(\bar{\beta})$ satisfying the inequality (4). For any $A \in U$, there exists a sequence of digits $(a_1(A), \dots, a_n(A))$ which is admissible in the $\bar{\beta}$ -expansion such that $A = (a_1(x) = a_1(A), \dots, a_n(x) = a_n(A))$. Since T is piecewise linear, we have $m(T^{n(A)}E) = \beta_{a_1(A)} \cdots \beta_{a_n(A)} m(E) = m(E)/m(A)$, for any $E \in \mathcal{B}$ in A . By this relation and Corollary 1, we obtain $P(T^{n(A)}E) \leq h(0)^2 \mu([0, 1])(P(E)/P(A))$. Thus we may set $q = h(0)^2 \mu([0, 1])$.

4. **Markov properties.** Let $x = (a_1(x), a_2(x), \dots)$ be a $\bar{\beta}$ -expansion of a real number x , $0 < x < 1$, then $Tx = (a_2(x), a_3(x), \dots)$, that is, T is a shift transformation of the stochastic process $(a_1(x), a_2(x), \dots)$, $0 < x < 1$, with a finite number of states. Since P is T -invariant the process is stationary.

Theorem 3. *Let $\bar{\beta}$ be rational of order r , then T is a stationary r -ple Markov chain. $r=0$ implies the independency of the process.*

Lemma 2. *Let $\bar{\beta}$ be rational of order r and let n be any non-negative integer. Then for any sequence of digits $(c_1, c_2, \dots, c_{n+r})$ which is admissible in the $\bar{\beta}$ -expansion, we have*

$$(5) \quad m((c_{n+1}, c_{n+2}, \dots, c_{n+r})) = \beta_{c_1} \beta_{c_2} \cdots \beta_{c_n} m((c_1, c_2, \dots, c_{n+r}))$$

where $(c_1, c_2, \dots, c_k) = (a_1(x) = c_1, a_2(x) = c_2, \dots, a_k(x) = c_k)$.

Proof. If $n=0$, then the relation (5) is trivial. Let $n \geq 1$. We suppose that (5) holds for $n-1$. Then we have

$$m((c_2, c_3, \dots, c_{n+r})) = \beta_{c_2} \beta_{c_3} \cdots \beta_{c_n} m((c_{n+1}, c_{n+2}, \dots, c_{n+r})).$$

Therefore, we must prove

$$(6) \quad m((c_2, c_3, \dots, c_{n+r})) = \beta_{c_1} m((c_1, c_2, \dots, c_{n+r}))$$

for any admissible sequence $(c_1, c_2, \dots, c_{n+r})$. Here (6) holds obviously

for $c_1=0, 1, \dots, N-1$. Thus it remains to show that (6) holds for $c_1=N$. To do this it suffices to prove

$$(7) \quad (c_2, c_3, \dots, c_{n+r}) \subset [0, T1).$$

Since $\bar{\beta}$ is rational of order r , $T1$ is an endpoint of an interval $(c'_1, c'_2, \dots, c'_k)$ of length $k \geq r$. So we have

$$(c_2, c_3, \dots, c_{n+r}) \subset [0, T1) \quad \text{or} \quad (c_2, c_3, \dots, c_{n+r}) \subset [T1, 1).$$

But the last relation contradicts the admissibility of the sequence $(N, c_2, \dots, c_{n+1})$. Thus we have the relation (7). By induction the lemma is proved.

Proof of Theorem 1. By Lemma 2, we have

$$\begin{aligned} m(a_{n+r+1}(x)=c_{n+r+1}; a_1(x)=c_1, \dots, a_{n+r}(x)=c_{n+r}) \\ = \frac{m((c_{n+1}, c_{n+2}, \dots, c_{n+r+1}))}{m((c_{n+1}, c_{n+2}, \dots, c_{n+r}))} = Q(c_{n+1}, \dots, c_{n+r+1}) \end{aligned}$$

where Q is a constant which depends only on the admissible sequence $(c_{n+1}, c_{n+2}, \dots, c_{n+r+1})$. Since $\bar{\beta}$ is rational of order r , $h(x)$ is constant on every interval $(c_1, c_2, \dots, c_k)$ of length $k \geq r$. Then, we have

$$P(a_{n+r+1}(x)=c_{n+r+1}; a_1(x)=c_1, \dots, a_{n+r}(x)=c_{n+r}) = Q(c_{n+1}, \dots, c_{n+r+1})$$

for any admissible sequence $(c_1, c_2, \dots, c_{n+r+1})$.

References

- [1] Rényi, A.: Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hung., **8**, 477-493 (1957).
- [2] Parry, W.: On the β -expansions of real numbers. Acta Math. Sci. Hung., **11**, 401-416 (1960).
- [3] Cigler, J.: Ziffernverteilung in ϑ -adischen Büchen. Math. Zeit., **75**, 8-13 (1961).
- [4] Rohlin, V. A.: Exact endomorphisms of a Lebesgue space. Izv Akad. Nauk SSSR, **25**, 499-530 (1961). Amer. Math. Soc. Transl., **39** (2), 1-36 (1964).