256. Pseudo-differential Operators in the Theory of Hyperfunctions

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In this note we first give the definition of the sheaf of pseudo-differential operators, the notion of which is originally due to Sato. The sheaf is defined using Sato's theory of the sheaf $\mathcal{C}$. For the definition and the main properties of the sheaf $\mathcal{C}$ we refer the reader to Sato [2].

Secondly we develop the theory of pseudo-differential operators of finite type and discuss some applications to the theory of linear partial differential operators.

This note is a summary of our forthcoming paper in which the details will be given.

1. Let f be a real analytic mapping from an n -dimensional real analytic manifold N to an m -dimensional real analytic manifold M . The mapping f defines a natural homomorphism of vector bundles: $N \times_M T^*M \rightarrow T^*N$, where T^*M and T^*N denote the cotangent bundle of M and that of N , respectively, and $N \times_M T^*M$ denotes the fibre product of T^*M and N over M . We denote its kernel by T_N^*M , which is not a vector bundle in general, and define S_N^*M to be $(T_N^*M - N)/\mathbb{R}^+$. ($\mathbb{R}^+$ denotes the multiplicative group of positive numbers.) S_N^*M is a closed subset of $N \times_M S^*M$, whose fibres over N are spheres. The $f^*: N \times_M T^*M \rightarrow T^*N$ induces a projection $\rho: N \times_M S^*M - S_N^*M \rightarrow S^*N$. We denote by $\tilde{\omega}$ the natural projection $N \times_M S^*M - S_N^*M - S^*M$.

Let $\mathcal{A}_M$ be the sheaf of real analytic functions and let ν_M be the sheaf of densities with real analytic coefficients, which becomes an invertible $\mathcal{A}_M$ -Module. By means of the theory of $\mathcal{C}$, we can define the following two fundamental homomorphisms, corresponding to the substitution and integration along fibre.

$$\begin{aligned} f^*: \rho_! \tilde{\omega}^{-1} \mathcal{C}_M &\rightarrow \mathcal{C}_N, \\ f_*: \tilde{\omega}_! \rho^{-1} (\mathcal{C}_N \otimes_{\mathcal{A}_N} \nu_N) &\rightarrow \mathcal{C}_M \otimes_{\mathcal{A}_M} \nu_M. \end{aligned}$$

Remark. If $f: X \rightarrow Y$ is a continuous map and $\mathcal{F}$ is a sheaf on X , $f_!(\mathcal{F})$ is a direct image with proper support, that is, $\Gamma(U, f_!(\mathcal{F}))$

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$=\{s \in \Gamma(f^{-1}U; \mathcal{F}); \text{supp}(s) \text{ is proper over } U\}$, for every open subset U of Y .

We identify N with the submanifold of $N \times M$ by the diagonal mapping. The projection $T^*(N \times M) \rightarrow N \times T^*M$ induces an isomorphism $S_N^*(N \times M) \rightarrow N \times_M S^*M$, by which we identify $S_N^*(N \times M)$ and $N \times_M S^*M$. In general, if $f: X \rightarrow Y$ is a fibre space with sphere as fibre, we denote by a the antipodal mapping of X , which maps $x \in X$ to its antipodal point x^a . For any sheaf $\mathcal{F}$ over X , we denote by $\mathcal{F}^a$ the direct image of $\mathcal{F}$ by a .

Definition 1. We define the sheaf $\mathcal{P}_f$ of pseudo-differential operators over f by $\mathcal{P}_f = \Gamma_{S_N^*(N \times M)}(\mathcal{C}_{N \times M} \otimes_{\mathcal{A}_M} v_M^a)$. Since the support of $\mathcal{P}_f$ is in $S_N^*(N \times M)$, we consider $\mathcal{P}_f$ as a sheaf on $N \times_M S^*M$.

If $M=N$ and f is the identity map, $\mathcal{P}_f$ is denoted by $\mathcal{P}_M$ or by $\mathcal{P}$ and a section of $\mathcal{P}_M$ is said to be pseudo-differential operator on S^*M . There are two fundamental bilinear homomorphisms

$$\begin{aligned} \rho_!(\mathcal{P}_f|_{N \times_M S^*M - S_N^*M} \times \tilde{\omega}^{-1}\mathcal{C}_M) &\rightarrow \mathcal{C}_N \\ \tilde{\omega}_!(\rho^{-1}\mathcal{C}_N \otimes_{v_N} \times \mathcal{P}_f^a|_{N \times_M S^*M - S_N^*M}) &\rightarrow \mathcal{C}_M \otimes_{v_M}. \end{aligned}$$

Roughly speaking, the first bilinear homomorphism is defined by $(K(y, x)dx, \varphi(x)) \mapsto \int K(y, x)\varphi(x)dx$ and, the latter by $(\psi(y)dy, K(y, x)dx) \mapsto \left[\int \psi(y)K(y, x)dx \right] dy$.

If $f: N \rightarrow M$ and $g: M \rightarrow L$ are two real analytic mappings, we can define the composition homomorphism of pseudo-differential operators over g and pseudo-differential operators over f . Let $p_1: N \times_M S^*L \rightarrow N \times_M S^*M$ and $p_2: N \times_M S^*L \rightarrow M \times_L S^*L$ be the projections deduced from $\tilde{\omega}: M \times_L S^*L \rightarrow S^*M$ and $f: N \rightarrow M$. Then the composition of pseudo-differential operators is described by the following bilinear homomorphism: $p_1^{-1}\mathcal{P}_f \times p_2^{-1}\mathcal{P}_g \rightarrow \mathcal{P}_{g \circ f}$.

Now, consider the case where $N=M$ and f is the identity. In this case $\mathcal{P}$ has a structure of sheaf of rings by the above composition homomorphism, which reduces to a sheaf bilinear homomorphism $\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ since S_N^*M is empty. By the operations $\mathcal{P} \times \mathcal{C} \rightarrow \mathcal{C}$ and $(\mathcal{C} \otimes v) \times \mathcal{P}^a \rightarrow \mathcal{C} \otimes v$, $\mathcal{C}$ may be considered as a left $\mathcal{P}$ -Module, and $\mathcal{C} \otimes v$ as a right $\mathcal{P}$ -Module.

2. We next study a special class of pseudo-differential operators which is easy to manipulate. In the following we fix some local coordinate system and identify M with a domain Ω in $\mathbf{R}^n$ and S^*M with $\Omega \times S^{n-1}$. We denote a point in S^*M by (x, η) , where η is a homogeneous coordinate of S^{n-1} .

Definition 2 (Pseudo-differential operators of finite type). We call a pseudo-differential operator P to be of finite type when its kernel function $K(x, x')$ is represented near (x_0, η_0) as $\int \sum_{j \leq -m} p_j(x, \eta) \Phi_j(\langle x - x', \eta \rangle + i0) \omega(\eta)$ where $p_j(x, \eta)$ and $\Phi_j(\tau)$ satisfy the following conditions (i) $\sim$ (iii), and j runs over the integers. $\omega(\eta)$ is the volume element

$$\sum_{j=1}^n (-1)^{j-1} \eta_j d\eta_1 \wedge \cdots \wedge d\eta_{j-1} \wedge d\eta_{j+1} \wedge \cdots \wedge d\eta_n:$$

$$(i) \quad \Phi_j(\tau) = \begin{cases} \frac{(-1)^j (n-j-1)!}{(-2\pi\sqrt{-1})^n} \cdot \frac{1}{\tau^{n-j}} & (j < n) \\ \frac{-1}{(2\pi\sqrt{-1})^n (j-n)!} \tau^{j-n} \log \tau & (j \geq n) \end{cases}$$

(ii) $p_j(x, \eta)$ is real analytic near (x_0, η_0) and positively homogeneous of degree $(-j)$ with respect to η , that is, $p_j(x, c\eta) = c^{-j} p_j(x, \eta)$ ($c > 0$).

(iii) There exist some complex neighbourhood V of (x_0, η_0) and some constant c such that the following estimate (*) holds;

$$(*) \quad \sup_{(z, \zeta) \in V} |p_j(z, \zeta)| \leq j! c^j \text{ for sufficiently large } j.$$

Definition 3. Let P be a pseudo-differential operator of finite type whose kernel function has the above form. Then we define the order of P by $(-1) \times \min \{j | p_j(x, \eta) \neq 0\}$. If P is of order m , then we define the principal symbol of P to be $p_{-m}(x, \eta)$.

Afterwards we sometimes write P as $\sum_{j \geq -m} p_j(x, D_x)$.

Proposition 4. Let the kernel function of P be given by

$$\int \sum_{k \leq -m} a_k(x, x', \eta) \Phi_k(\langle x - x', \eta \rangle + i0) \omega(\eta),$$

where $a_k(x, x', \eta)$ is real analytic near (x_0, x_0, η_0) , is positively homogeneous of degree $(-k)$ with respect to η and satisfies the estimate $\sup_{(z, z', \zeta) \in V} |a_k(z, z', \zeta)| \leq k! c^k$ ($k \gg 0$), for some constant c and for some complex neighbourhood V of (x_0, x_0, η_0) . Then the kernel function of P has the form stated in Definition 2, that is $\int \sum p_j(x, \eta) \Phi_j(\langle x - x', \eta \rangle + i0) \omega(\eta)$,

where

$$p_j(x, \eta) = \sum_{j=k+|\alpha|} \frac{1}{\alpha!} D_x^\alpha D_\eta^\alpha a_k(x, x', \eta)|_{x'=x}.$$

This proposition is proved using the Taylor expansion of a_k with respect to $(x - x')$ and Cauchy's integral formula.

Corollary 1. The property that a pseudo-differential operator P is of finite type is independent of the choice of the local coordinate system.

Corollary 2. The principal symbol of P is independent of the choice of the local coordinate system, as a homogeneous function on S^*M .

Theorem 5. *Let P and Q be pseudo-differential operators of finite type expressed in the form $\sum p_j(x, D_x)$ and $\sum q_k(x, D_x)$, respectively. Then the composed operator $R=PQ$ is also of finite type and if we write $R=\sum r_l(x, D_x)$, then*

$$r_l(x, \eta) = \sum_{l=j+k+|\alpha|} \frac{1}{\alpha!} D_\eta^\alpha p_j(x, \eta) D_x^\alpha q_k(x, \eta).$$

To prove this theorem we use the following formula which holds as an identity of sections of the sheaf $\mathcal{C}$;

$$\begin{aligned} & \int (\langle x, \xi \rangle + i0)^\lambda (t - \langle x, \eta \rangle + i0)^\mu dx \\ &= \frac{(-2\pi i)^n \Gamma(-\lambda - \mu - n)}{\Gamma(-\lambda) \Gamma(-\mu)} (t + i0)^{\lambda + \mu + n} \delta_\lambda(\xi; \eta), \end{aligned}$$

where $\delta_\lambda(\xi; \eta)$ is a δ -function concentrated on $\xi = c\eta$ ($c > 0$), homogeneous of degree λ in ξ and of degree $(-n - \lambda)$ in η .

Theorem 6. *Let P be a pseudo-differential operator of finite type and $p_{-m}(x, \eta)$ be its principal symbol. Then $\text{supp Ker}_\mathcal{C} P$ and $\text{supp Coker}_\mathcal{C} P$ are both contained in $\{(x, \eta) \in S^*M \mid p_{-m}(x, \eta) = 0\}$, where $\text{Ker}_\mathcal{C} P$ is by definition the solution sheaf of the pseudo-differential operator P which operates on the sheaf $\mathcal{C}$, and $\text{Coker}_\mathcal{C} P$ denotes the cokernel of P .*

This can be proved by the majorant method using the calculation of Theorem 5.

Remark. This theorem was first found by Sato [1] for partial differential operators and is called Sato's fundamental theorem.

Theorem 7. *Let P and Q be partial differential operators of order m with simple characteristics and $P_{-m}(x, \eta) = Q_{-m}(x, \eta)$ is a polynomial of (x, η) (for some local coordinate system). Then we can find locally some invertible pseudo-differential operators of finite type R and S such that $RQ = PS$.*

Remark. The assumption that P and Q are partial differential operators and that $P_{-m}(x, \eta) = Q_{-m}(x, \eta)$ is a polynomial is very unsatisfactory. We hope that the assumption can be relaxed at least to the condition that P and Q are pseudo-differential operators of finite type of order m with simple characteristics and $P_{-m} = Q_{-m}$. In fact, we can find R and S formally in that case but we have not yet proved the estimate (*) in Definition 2.

Sketch of the proof of Theorem 7.

We can assume that $Q(x, \eta) = Q_{-m}(x, \eta)$ without loss of generality. If $P_{-m}(x, \eta) \neq 0$ then Theorem 6 proves this theorem. So we can assume $\text{grad}_\eta P_{-m} \neq 0$ by the assumption of simple characteristics. (In fact, it is sufficient to assume $\text{grad}_{(\eta, x)} P_{-m}(x, \eta) = (\text{grad}_\eta P_{-m}(x, \eta), \text{grad}_x P_{-m}(x, \eta))$ is not parallel to $(0, \eta)$ if $P_{-m}(x, \eta) = 0$ instead of the

above assumption.) Assuming $R=S=\sum_{j \geq 0} r_j(x, D_x)$ we obtain successively the first order partial differential equations of the following type:

$$\begin{aligned} &\langle \text{grad}_\eta P_{-m}(x, \eta), \text{grad}_x r_j \rangle - \langle \text{grad}_\eta r_j, \text{grad}_x P_{-m} \rangle \\ &+ P_{-m+1}(x, \eta) r_j(x, \eta) = \rho_j(x, \eta), \end{aligned}$$

where $\rho_0=0$ and $\rho_j(x, \eta) (j \geq 1)$ is determined by $\{r_k\}_{k=0, \dots, j-1}$. By the assumption on $\text{grad}_\eta P_{-m}$ we can find some non-characteristic surface $T=\{t(x, \eta)=0\}$ and give the initial condition of r_j on T by 1 for $j=0$ and by 0 for $j \geq 1$. Using Euler's identity for homogeneous functions we can find $\{r_j(x, \eta)\}$ satisfying the conditions of Definition 2.

References

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