30. On Vector Measures. II

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In [4] we have proved the following theorem. Let $S$ be a set, $R$ a semi-tribe (δ-ring) of subsets of $S$, $X$ a normed space and $m: R \to X$ a vector measure. Then there exists a finite non-negative measure $\nu$ on $R$ such that

1. for any $A \in R$ and any number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that $B \in R, B \subseteq A$ and $\nu(B) < \delta \Rightarrow \|m(B)\| < \varepsilon$

2. $\nu(E) \leq \sup \{\|m(A)\| ; A \subseteq E, A \in R\}$ for $E \in R$ ([4] Theorem 1).

The purpose of this paper is to point out some properties of regular vector measures by using this theorem. These properties were proved earlier (Dinculeanu [1] § 16, Theorem 3, Corollaries 1–4) for vector measures with finite variation, but we shall drop this condition and we shall consider the necessary and sufficient condition for the extension of a regular, finitely additive set function from some clan to a wider class of subsets (cf. Theorem 3). And Corollary 1 is the extension of Dinculeanu's and Kluvanek's result ([2] Theorem 5).

3. Regular vector measures. Suppose that $S$ be a locally compact, Hausdorff space and $X$ a Banach space.

Definition 3. Let $R$ be a clan (ring) of subsets of $S$. A set function $m: R \to X$ is called regular if for every $A \in R$ and every number $\varepsilon > 0$ there exists a compact set $K \subseteq A$ and an open set $G \supseteq A$ such that for every $A' \in R$ with $K \subseteq A' \subseteq G$ we have $m(A) - m(A') < \varepsilon$.

Definition 4. Let $m: R \to X$ be a set function and $\mu$ a non-negative measure on $R$. $m$ is $\mu$-absolutely continuous if for every $A \in R$ and every number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that for every $B \in R$ with $B \subseteq A$ and $\mu(B) < \delta$ we have $\|m(B)\| < \varepsilon$.

Lemma 2. Let $R$ be semi-tribe of subsets of $S$ which has the following conditions

(*) If $m: R \to X$ is a regular vector measure, then there exists a finite non-negative measure $\nu$ on $R$ such that

1. $m$ is $\nu$-absolutely continuous.
2. $\nu(E) \leq \sup \{\|m(A)\| ; A \subseteq E, A \in R\}$ for $E \in R$.
3. $\nu$ is regular.


Theorem 3. Let $R$ be a clan which has the following conditions
for every compact set $K$ and for every open set $G$ such that $G \supset K$
for every $A \in R$ such that $K \subset A \subset G$.

(\text{*}) for every $A \in R$ there exists an $A' \in R$ such that $A \subset \text{Int}A'$.

Then every regular and finitely additive set function $m; R \to X$ can be extended uniquely to a regular vector measure $m_i$, on the semi-tribe $\varphi$ generated by $R$ if and only if there exists a finite, non-negative, regular measure $\nu$ on $R$ such that $m$ is $\nu$-absolutely continuous. In this case, $m$ becomes countably additive.

**Proof.** The necessity is immediate by Lemma 2.

Sufficiency. By Dinculeanu ([1] § 16, Theorem 2, Corollary 2) $\nu$ can be extended uniquely to a finite, non-negative regular measure $\nu_i$ on $\varphi$. For any $A \in \varphi$ there exists a $E \in R$ with $E \supset A$ (Dinculeanu [1] § 1, Proposition 10, corollary). By $\nu$-absolute continuity of $m$, for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that $B \in R$, $B \subset E$ and $\nu_i(B) = \nu(B) < \delta \Rightarrow ||m(B)|| < \varepsilon$. Hence $B$, $C \in R$, $B \subset C$, $C \subset E$ and $\nu_i(B \cup C) = \nu(B \cup C) < \delta$.

$$||m(B) - m(C)|| = ||m(B - C) - m(C - B)|| 
\leq ||m(B - C)|| + ||m(C - B)|| < 2\varepsilon$$

Since $\nu_i$ is regular, for above $A$ and $\delta$ there exists a compact set $K \subset A$ and an open set $G \supset A$ such that $B \in R$, $B \subset G - K$ implies $\nu_i(B) < \delta$.

By (\text{*}) there exists a $B \in R$ with $K \subset B \subset G$. Therefore $B \cap E \in R$ and $A \cup (B \cap E) \subset G - K$ implies $\nu_i(A \cup (B \cap E)) < \delta$. Now we take $B_1 \in R$ and $B_2 \in R$ such that $\nu_i(A \cup (B_1 \cap E)) < \frac{1}{2} \delta$

and

$$\nu_i(A \cup (B_2 \cap E)) < \frac{1}{2} \delta$$

Then

$$\nu((B_1 \cap E) \cup (B_2 \cap E)) = \nu_i((B_1 \cap E) \cup (B_2 \cap E)) 
\leq \nu_i(A \cup (B_1 \cap E)) + \nu_i(A \cup (B_2 \cap E)) < \frac{1}{2} \delta$$

Hence we have $||m(B_1 \cap E) - m(B_2 \cap E)|| < 2\varepsilon$. Since $X$ is complete space, we have $m_E(A) = \lim_{\nu_i(A \cup (B \cap E)) \to 0} m(B \cap E)$. In particular, $m_E(A) = m(A)$ for $A \in R$. The uniqueness of $m_E$ is clear.

Next we shall prove that $m_E$ is independent on $E(\supset A)$. For any $F \in R$ with $A \subset F \subset E$ and every number $\varepsilon > 0$, there exists $\delta_1 = \delta(\varepsilon, E)$ and $\delta_2 = \delta(\varepsilon, F) > 0$. We set $\delta = \min(\delta_1, \delta_2)$. We take $B_1$, $B_2 \in R$ such that $\nu_i(A \cup (B_1 \cap E)) < \frac{1}{2} \delta$ and $\nu_i(A \cup (B_2 \cap E)) < \frac{1}{2} \delta$. Then $\nu_i((B_1 \cap E) \cup (B_2 \cap F)) < \delta \leq \delta_1$ and $B_2 \cap F \subset E$.

It follows that $||m(B_1 \cap E) - m(B_2 \cap F)|| < 2\varepsilon$. Therefore $||m_E(A) - m_F(A)|| \leq 2\varepsilon$. Since $\varepsilon$ is arbitrary, we have $m_E(A) = m_F(A)$. For every $F \in R$ with $F \supset A$, $m_F(A) = m_{E \cap F}(A) = m_E(A)$.

If we put $m_i(A) = m_F(A)$ ($A \subset E \subset F$), we have

(\text{i}) $m_i$ is finitely additive.
(ii) $m_1$ is $\nu_1$-absolutely continuous.
(iii) $m_1$ is regular.
(iv) $m_1$ is countably additive.

Since (i) is clear (see Kluvanek [3] Theorem 1), (iii) is clear by (**) and (ii), and (iv) is clear by (ii), it only remains to prove (ii): For any $E \in \mathfrak{E}$ there exists a $F \in \mathcal{R}$ with $F \supset E$. From $\nu$-absolute continuity of $m$ for every number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, F) > 0$ such that $A \subset F$ $A \in \mathcal{R}$ and $\nu(A) < \delta$ implies $\|m(A)\| < \frac{1}{2} \varepsilon$. Let $B \in \mathfrak{E}$ be a set such that $B \subset E$ and $\nu(B) < \delta$. We put $\delta_1 = \nu(B)$. Then from the definition of $m_1(B)$, there exists a number $\delta_2 = \delta(\varepsilon, E) > 0$ such that $\nu_1(B \Delta (B_1 \cap F)) < \delta_1$ implies $\|m_1(B) - m(B_1 \cap F)\| < \frac{1}{2} \varepsilon$. We put $\delta_2 = \min (\delta - \delta_1, \delta_3)$. Let $B_1 \in \mathcal{R}$ be a set with $\nu_1(B \Delta (B_1 \cap F)) < \delta_2$. Then

$$\|m_1(B) - m(B_1 \cap F)\| < \frac{1}{2} \varepsilon.$$  

$$|\nu_1(B) - \nu(B_1 \cap F)| = |\nu_1(B - B_1 \cap F) - \nu_1(B_1 \cap F - B)|$$

$$\leq \nu_1(B \Delta (B_1 \cap F)) < \delta_2 \leq \delta - \delta_1$$

so $\nu(B_1 \cap F) < \nu_1(B) + \delta - \delta_1 = \delta$. Therefore $\|m(B_1 \cap F)\| < \frac{1}{2} \varepsilon$. Thus we have $\|m_1(B)\| \leq \|m_1(B) - m(B_1 \cap F)\| + \|m(B_1 \cap F)\| < \varepsilon$. The uniqueness of $m_1$ is clear by the uniqueness of $m_1$. Q.E.D.

Denote by $\mathfrak{K}_0$ the semi-tribe of the relatively compact Baire sets, by $\mathfrak{B}$ the semi-tribe of the relatively compact Borel sets and by $\mathfrak{K}_0$ the clan generated by the compact sets with are $G_0$.

**Corollary 1.** Let $\mathcal{R}$ be a clan such that $\mathfrak{K}_0 \subset \mathcal{R} \subset \mathfrak{B}$. every regular and finitely additive set function $m: \mathcal{R} \to X$ can be extended uniquely to a regular Borel measure $m_1; \mathfrak{B} \to X$ if and only if there exists a finite, non-negative, regular measure $\nu$ on $\mathcal{R}$ such that $m$ is $\nu$-absolutely continuous. In this case $m$ becomes countably additive.

**Proof.** By Dinculeanu ([1] §14, Propositions 11 and §15, Lemma 1) $\mathcal{R}$ is satisfied the conditions $(*)$, $(**)$(*) of Theorem 3. Then we can prove in the same way as the proof of Theorem 3.

If we put $\mathcal{R} = \mathfrak{B}_0$ Then we have the following result.

**Corollary 2.** Every Baire measure $m; \mathfrak{B}_0 \to X$ can be extended uniquely to a regular Borel measure $m_1; \mathfrak{B} \to X$ (Dinculeanu and Kluvanek [2] Theorem 5).

**References**

