79. On the Existence of a Potential Theoretic Measure with Infinite Norm

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Introduction. Let $R^m$ be the $m$-dimensional Euclidian space and $\phi(x, y)$ a lower semi-continuous function from $R^m \times R^m$ into $[0, +\infty]$. The $\phi$-potential of a positive Radon measure $\mu$ in $R^m$ is defined by

$$\phi\mu(x) = \int \phi(x, y)d\mu(y).$$

In the case that there exists at least such a positive measure $\nu$ that the support $S\nu$ is compact and the potential $\phi\nu(x)$ is continuous in the whole space $R^m$, we can consider the following classes of measures:

- $F(\phi) = \{\nu; \nu \geq 0, S\nu \text{ compact and } \phi\nu(x) \text{ continuous in } R^m\}$,
- $\mathcal{G}(\phi) = \{\mu; \mu \geq 0 \text{ and } \int \phi\mu d\nu < +\infty \text{ for any } \nu \in F(\phi)\}$.

The aim of this paper is to answer affirmatively for a question posed by G. Anger [1]: Let $\phi_N(x, y)$ be the Newtonian kernel defined in $R^m$ ($m \geq 3$). Is there a measure $\mu \in \mathcal{G}(\phi_N)$ with infinite norm? Moreover we study the same problem in case of $\alpha$-kernel $\phi_\alpha(x, y)$.

1. Existence of a measure $\mu \in \mathcal{G}(\phi_N)$ with infinite norm.

The Newtonian kernel $\phi_N(x, y)$ in $R^m$ ($m \geq 3$) is defined by

$$\phi_N(x, y) = |x - y|^{2-m},$$

where $|x - y|$ denotes the distance between two points $x$ and $y$ in $R^m$. Let $B_{a,r}$ be the closed ball with the center $a$ and the radius $r$ and $S_{a,r}$ the surface of the ball $B_{a,r}$. We introduce the class of measures

$$S = \{\lambda; \text{spherical distribution with uniform density}\}.$$

Especially the spherical distribution with uniform density on $S_{a,r}$ is denoted by $\lambda_{a,r}$. It is well known that $S$ is a non empty subset of $F(\phi_N)$. Let us recall following potential theoretic principles,

- Maximum principle: If it holds that, for a constant $V$, $\phi\nu(x) \leq V$ on the support $S\nu$ of a positive measure $\nu$, then we have the same inequality in the whole space.
- Domination principle: If it holds that, for a positive measure $\nu$ and an energy finite positive measure $\mu$, $\phi\mu(x) \leq \phi\nu(x)$ on the support $S\mu$, then we have the same inequality in the whole space.

Lemma 1. For a given positive measure $\mu$, the mutual energy

$$\int \phi_N \mu d\nu$$

is finite for any $\nu \in F(\phi_N)$ if $\int \phi_N \mu d\lambda$ is finite for any $\lambda \in S$. 

Proof. It is sufficient to show that, for any \( \nu \in \mathcal{F}(\phi_N) \), we can choose such a suitable measure \( \lambda \in \mathcal{S} \) that \( \phi_N \lambda(x) \geq \phi_N \nu(x) \) in the whole space \( \mathbb{R}^m \). The Newtonian kernel satisfying the maximum principle, the potential \( \phi_N \nu(x) \) of a measure \( \nu \) attains the maximum on the compact support \( S\nu \). Let \( \lambda_{a,r} \) be the spherical distribution with uniform density and the total mass \( M > 0 \). It is well known that

\[
\phi_N \lambda_{a,r}(x) = \begin{cases} 
\frac{M}{r^{m-2}} & \text{in } B_{a,r} \\
\frac{M}{|x-a|^{m-2}} & \text{otherwise.}
\end{cases}
\]

(\*) Consequently, choosing a suitable center \( a \), a radius \( r \) and a sufficiently large total mass \( M \), we can pick up such a measure \( \lambda_{a,r} \), that the corresponding ball \( B_{a,r} \) contains the compact support \( S\nu \) and \( \phi_N \lambda_{a,r}(x) \geq \phi_N \nu(x) \) on \( S\nu \). The Newtonian kernel satisfying the domination principle, it follows that

\[
\phi_N \mu_{a,r}(x) \geq \phi_N \nu(x) \quad \text{in the whole space } \mathbb{R}^m.
\]

Lemma 2. Let \( \mu_s \) be the measure

\[
\mu_s = \sum_{n=1}^{+\infty} n^{m-s} \mu_n \quad \text{for any real number } s \, (3 < s \leq 1 + m),
\]

where \( \mu_n \) denotes a unit point mass on the sphere \( S_{0,n} \) with the center the origin \( 0 \) and the radius \( n \). Then \( \mu_s \) is a positive Radon measure with infinite norm and we have \( \int \phi_N \mu d\lambda < +\infty \) for any \( \lambda \in \mathcal{S} \).

Proof. It is obvious that \( \mu_s \) is a positive Radon measure and, owing to \( s - m \leq 1 \), we have

\[
\|\mu_s\| = \mu_s(\mathbb{R}^m) = \sum_{n=1}^{+\infty} n^{m-s} = +\infty \quad \text{for any } s.
\]

On account of (\*), we have, for a measure \( \lambda_{a,r} \) with the norm \( M \),

\[
\int \phi_N \mu_s d\lambda_{a,r} = \int \phi_N \lambda_{a,r} d\mu_s \\
= \int_{B_{a,r}} \phi_N \lambda_{a,r} d\mu_s + \int_{\mathbb{R}^m - B_{a,r}} \phi_N \lambda_{a,r} d\mu_s \\
= \sum_{|n-a| \leq r} \frac{M}{r^{m-2}} \cdot \frac{1}{n^{s-m}} \\
+ \sum_{|n-a| > r} \frac{M}{|n-a|^{m-2}} \cdot \frac{1}{n^{s-m}} \\
< +\infty,
\]

because the first summation of the right hand side is obviously finite and the second summation is the same order of \( \sum_{n=1}^{+\infty} n^{2-s} \).

By Lemmas 1 and 2, we have immediately the following theorem.

Theorem 1. The measure in Lemma 2

\[
\mu_s = \sum_{n=1}^{+\infty} n^{m-s} \mu_n
\]
is an element of $\mathcal{Q}(\mathcal{N})$ with infinite norm.

Remark. H. Cartan [2] characterised the class of measures $\mathcal{Q}(\mathcal{N})$: A measure $\mu$ is an element of $\mathcal{Q}(\mathcal{N})$ if and only if $\phi_N \mu(x) \neq +\infty$. The above theorem shows that there are infinitely many positive measures $\mu_s$ with infinite norm that $\phi_N \mu_s(x) \neq +\infty$.

2. Existence of a measure $\mu \in \mathcal{Q}(\mathcal{N})$ with infinite norm.

The $\alpha$-kernel $\phi_\alpha(x, y)$ in $\mathbb{R}^m$ is defined by

$$\phi_\alpha(x, y) = |x-y|^{\alpha-m}$$

where $\alpha$ is any real number such as $0 < \alpha < m$. O. Frostman [3] studied deeply the $\alpha$-potential and proved many remarkable theorems. Above all, we start from his following theorem: Given a closed region $F$ of which the boundary satisfies the Poincaré's condition, there exists a positive measure $\gamma$ with unit mass and supported by $F$ of which the potential $\phi_\alpha \gamma(x)$ is a positive constant $V$ on $F$ and is continuous in $\mathbb{R}^m$. We shall say such a measure the equilibrium measure on $F$. This shows that $\mathcal{F}(\phi_\alpha)$ is not empty and we can consider the class of measures

$$\mathcal{U}(\phi_\alpha) = \{\text{Equilibrium measure on all balls in } \mathbb{R}^m\}.$$  

Especially the equilibrium measure on the ball $B_{a, r}$ is denoted by $\gamma_{a, r}$.

Lemma 3. For a given positive measure $\mu$, the mutual energy $\int \phi_\alpha \mu \nu$ is finite for any $\nu \in \mathcal{F}(\phi_\alpha)$ if $\int \phi_\alpha \mu \nu$ is finite for any $\gamma \in \mathcal{U}(\phi_\alpha)$.

Proof. By the analogous way in the demonstration of Lemma 1, we can choose such a measure $\gamma \in \mathcal{U}(\phi_\alpha)$ that, for a suitable positive number $t$, $\phi_\alpha \gamma(x) \geq t \phi_\alpha \nu(x)$ in $\mathbb{R}^m$, because the $\alpha$-kernel also satisfies the maximum and domination principles.

Lemma 4. For any index $\alpha$ such as $0 < \alpha \leq 2$, any positive number $t$ and any measure $\gamma \in \mathcal{U}(\phi_\alpha)$, there exists such a spherical distribution with uniform density $\lambda$ that $\phi_\alpha \lambda(x) \geq t \phi_\alpha \gamma(x)$ in $\mathbb{R}^m$.

Proof. For any index $\alpha$ such as $0 < \alpha \leq 2$, the $\alpha$-potential of a positive measure $\nu$ is subharmonic in the complementary set of the support $S \nu$. On the other hand, the Newtonian potential of a positive measure is superharmonic in $\mathbb{R}^m$. So, in order to prove this lemma, it is sufficient to show that, for any positive number $t$ and any measure $\gamma_{a, r} \in \mathcal{U}(\phi_\alpha)$, there exists a suitable measure $\lambda \in S$ that $\phi_\alpha \lambda(x) \geq t \phi_\alpha \gamma_{a, r}(x)$ on the sphere $S_{a, r}$, the boundary of $B_{a, r}$. Owing to (*) and choosing $\lambda_{a, r} \in S$ with a sufficiently large total mass $M$, we can make the value of $\phi_\alpha \lambda_{a, r}(x)$ on $S_{a, r}$ larger than that of $t \phi_\alpha \gamma_{a, r}(x)$ on the same sphere.

By Lemmas 2, 3 and 4, we have the following theorem.

Theorem 2. The measure in Lemma 2

$$\mu_s = \sum_{n=1}^{+\infty} n^{t-m} \mu_n$$
is also an element of $L(\phi_a)$ ($0 < \alpha \leq 2$) with infinite norm.

References

