173. On a Conjecture of K. S. Williams

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1. Let p be a rational prime and n a positive integer ≥ 2 . We denote by $a_n(p)$ the least positive integral value of a which makes the polynomial $x^n + x + a$ irreducible (mod p). In a recent paper [3] K. S. Williams conjectured that for all $n \geq 2$ one has

$$(1) \quad \liminf_{p \rightarrow \infty} a_n(p) = 1,$$

and showed (among others) that (1) is true for $n=2$ and 3. In the present note we shall prove that (1) is true for $n=4, 6, 9, 10$ and for all primes $n \equiv 1 \pmod{3}$. However, it is immediately clear that (1) is not true for some (in fact, infinitely many) values of n . Indeed, the polynomial $x^n + x + 1$ is irreducible in $Z[x]^*)$ if and only if $n=2$ or $n \not\equiv 2 \pmod{3}$, and for $n \equiv 2 \pmod{3}$ $x^n + x + 1$ has the obvious factor $x^2 + x + 1$ (cf. [2]). Thus, we can show that for $n=5$

$$(2) \quad \liminf_{p \rightarrow \infty} a_5(p) = 3$$

and for $n=8$

$$(3) \quad \liminf_{p \rightarrow \infty} a_8(p) = 2.$$

2. Our foundation is on the following important theorem due to F. G. Frobenius [1].

Theorem. *Let $f(x)$ be a square-free polynomial (i.e. a polynomial with non-zero discriminant) of degree $n \geq 1$ in $Z[x]$, and let $d_1, \dots, d_r$ ($r \geq 1$) be positive integers with $d_1 + \dots + d_r = n$. Then, if the Galois group of $f(x)$, as a permutation group on n letters, contains a permutation which is decomposed as the product of r cycles of length $d_1, \dots, d_r$, there are infinitely many primes p such that we have*

$$(4) \quad f(x) \equiv f_1(x) \cdots f_r(x) \pmod{p},$$

where $f_1(x), \dots, f_r(x)$ are polynomials of $Z[x]$, each irreducible (mod p), of degree $d_1, \dots, d_r$, respectively.

In fact, it is proved in [1] that the Dirichlet density of prime numbers p for which (4) holds equals the number of permutations in the Galois group of $f(x)$ that have r cycles of length $d_1, \dots, d_r$, divided by the order of the group.

By virtue of this theorem, a simple and well-known argument on the reduction (mod p) of the Galois group of $f(x)$ will show that the

*) We denote by Z , as usual, the ring of rational integers.

existence of infinitely many primes p satisfying (4) is equivalent to the existence of at least one such prime p (not dividing the discriminant of $f(x)$).

The following result is a particular case of the above theorem.

Corollary. *Let $f(x)$ be a polynomial of degree $n \geq 1$ in x with coefficients in Z . If the Galois group of $f(x)$ contains a cycle of length equal to n , then there are infinitely many primes p for which $f(x)$ is irreducible (mod p) (so that $f(x)$ is necessarily irreducible in $Z[x]$).*

Another interesting consequence of the theorem of Frobenius is that if $f(x)$ is an irreducible polynomial of degree $n \geq 2$ in $Z[x]$, then there exists an infinity of primes p such that the congruence

$$(5) \quad f(x) \equiv 0 \pmod{p}$$

has no solution x in Z . On the other hand, it is not difficult to see that for an arbitrary non-constant polynomial $f(x)$ in $Z[x]$ there are infinitely many primes p for which the congruence (5) has solutions x in Z .

3. Now, we shall apply the corollary to the theorem of Frobenius, to the special polynomials $x^n + x + a$, $a \in Z$. The polynomial $x^n + x + 1$ is irreducible in $Z[x]$ for $n \equiv 1 \pmod{3}$. Hence, if $n \equiv 1 \pmod{3}$ is prime, then there are infinitely many primes p for which $x^n + x + 1$ is irreducible (mod p). In order to obtain the other results enunciated in § 1, it will suffice to find the least positive value of a and an appropriate prime number p such that $x^n + x + a$ is irreducible (mod p). Thus, the polynomial $x^n + x + 1$ is irreducible (mod 2) for $n = 2, 3, 4, 6, 9$ and 10. Hence, (1) holds true for these values of n . Next, we see that the polynomial $x^5 + x + 3$ is irreducible (mod 7), so that (2) holds. Finally, the polynomial $x^8 + x + 2$ is irreducible (mod 3) and (3) holds.

Also, we can argue for the case of $n = 6$ in the following way. We find that

$$x^6 + x + 1 \equiv (x+2)(x^2+2x+2)(x^3+2x^2+x+1) \pmod{3}$$

and

$$x^6 + x + 1 \equiv (x+2)(x^5+5x^4+4x^3+6x^2+2x+4) \pmod{7},$$

the factors on the right-hand side being irreducible to the respective moduli. It follows from this that the Galois group of $x^6 + x + 1$ is the symmetric group of degree 6. (Here, use was made of the elementary fact that, if a transitive permutation group on n letters contains a transposition and a cycle of length $n-1$, then the group coincides with the symmetric group of degree n .) Thus, the density of prime numbers p for which $x^6 + x + 1$ is irreducible (mod p) is equal to $1/6$.

Our method could of course be extended to treat the case of $n > 10$, unless we avoided the increasing complication with n in the reduction of the relevant polynomials with various moduli (except for primes $n \equiv 1 \pmod{3}$).

Added in proof (September 21, 1970). It is not difficult to see that the polynomial $x^n + x + a$, $a \in \mathbb{Z}$, is irreducible in $\mathbb{Z}[x]$ for $a=3$ and $n \geq 2$ (for $a=2$ and even $n \geq 2$). We thus have, in particular,

$$\liminf_{p \rightarrow \infty} a_n(p) = 3$$

for every odd prime $n \equiv 2 \pmod{3}$.

References

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