178. On the Adjoint Semigroups of Rings. I

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The fundamental notions, used in this paper, can be found in A. H. Clifford-G. B. Preston [1], N. Jacobson [3] and N. H. McCoy [4]. As it is well known, H. J. Hoehnke [2] has developed an interesting theory of a radical for semigroups with zero such that this radical is similar in certain sense to the Jacobson radical of rings (see [3]). Furthermore, H. Seidel [5] has introduced for semigroups, not necessarily having zero, the concept of right quasiregular elements, and he has proved [5] with the help of right quasiregular elements, that the Hoehnke radical of any semigroup S with zero coincides with the nil radical of S . (In addition see yet author's paper [7] on six further similar radicals of semigroups.) We recall, that an element $s \in S$ is right quasiregular in the semigroup S if and only if for arbitrary elements $t \in S$ and $u \in S$ nonnegative rational integers m and n there exist such that $s^m t = s^n u$ holds, eventually $s^0 t$ denoting the element t .

Some results on general radicals of semigroups with zero are discussed in author's paper [8].

A semigroup S having twosided unity element e is said to be *almost right quasiregular*, if $S = e \cup Q$ with $e \notin Q$ holds such that Q is a subsemigroup of S , and any element of Q is right quasiregular in S .

Furthermore, a semigroup S with both twosided unity element e and zero will be called *almost nil* (or *almost nilpotent*), if $S = e \cup N$ with $e \notin N$ holds, where N is a nil (or nilpotent, respectively) subsemigroup of S . Here N is said to have a bounded index m of nilpotency of elements, if a natural number m there exists such that $x^m = 0 \in N$ for any $x \in N$ holds.

Thirdly, a semigroup S with both twosided unity element e and zero will be called *almost trivial*, if $S = e \cup T$ with $e \notin T$ holds, where T is a subsemigroup of S such that all products xy for arbitrary $x \in T$ and $y \in T$ coincide with the zero element of S .

Therefore, any almost trivial semigroup is commutative.

All rings, considered here, will be associative. For any ring A , the elements of A form with respect to the circle operation $a \circ b = a + b - ab$ a semigroup S , which is called the *adjoint semigroup* of the ring A . Obviously the zero 0 of A is the twosided unity element of S . Furthermore an element e of A is a right zero of S if and only if e is

a right unity element of A . A ring A will be called H -ring, if its adjoint semigroup S is almost right quasiregular. H -rings can be considered as very strongly nonradical rings for the Jacobson radical [3], the adjoint semigroup of any Jacobson radical ring being a group. For results on the circle operation we refer the reader for instance to [3], [6] and [9].

The aim of this paper is to point out, without proof, the equivalence of thirteen conditions for adjoint semigroups of rings. Namely we have the following:

Theorem. *The following thirteen conditions for an adjoint semigroup S of a ring $A (\neq 0)$ are mutually equivalent:*

- (I) *S is almost right quasiregular.*
- (II) *S is almost right quasiregular with an idempotent element, which differs from the unity element of S .*
- (III) *S is almost right quasiregular with a left zero.*
- (IV) *S is almost right quasiregular with a right zero.*
- (V) *S is almost right quasiregular with zero.*
- (VI) *S is almost nil.*
- (VII) *S is almost nil with bounded index of nilpotency of elements.*
- (VIII) *S is almost nilpotent.*
- (IX) *S is almost nilpotent and commutative.*
- (X) *S is almost trivial.*
- (XI) *S is almost trivial and finite.*
- (XII) *S is the adjoint semigroup of the field of two elements.*
- (XIII) *S consists of two elements a and b with the multiplication:*

	a	b
a	a	a
b	a	b

Remark. We mention that the equivalence of conditions (VI) and (XII) is verified in author's paper [9].

References

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