203. **On Potent Rings. I**

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A ring \( R \) is said to be (right) potent iff every nonzero closed right ideal \( A \) of \( R \) is potent, that is, \( A^n \) is not zero for all positive integer \( n \). In [6], R. E. Johnson has investigated potent irreducible rings which are finite dimensional in the sense of Goldie [4], and obtained many interesting results. The aim of this paper is to generalize the Johnson’s work [6] to the case of the rings with infinite dimensions.

1. Definitions and notations.

Let \( R \) be an associative ring. A right ideal \( I \) of \( R \) is called closed if it has no proper essential extensions in \( R \) as right \( R \)-modules. Clearly the concept of closed right ideals of \( R \) coincides with the one of complemented right ideals in the sense of Goldie [4]. A right ideal \( E \) of \( R \) is called large if \( R \) is an essential extension of \( E \) (in symbols \( E \subsetneq R \)). A ring \( R \) is said to be (right) locally uniform if any nonzero right ideal of \( R \) contain a nonzero uniform right ideal. A right ideal \( A \) is uniform if \( A \) is an essential extension of every nonzero right ideal contained in \( A \). Clearly, if \( R \) is finite dimensional, then \( R \) is locally uniform. \( R \) is called countably dimensional if \( R \) has a direct sum of countable right ideals. The notation \( A'(A') \) is used for right (left) annihilator of a subset \( A \) of \( R \).

The set \( Z_r(R) = \{ x \in R | x' \text{ : large right ideal of } R \} \) is an ideal of the ring \( R \), which is called the right singular ideal. If \( Z_r(R) = 0 \), then the each right ideal \( A \) has a unique maximal essential extension \( A^* \) in \( R \). The set \( L_r^*(R) (= L_r^*) \) of closed right ideals is a complete complemented modular lattice under the inclusion. If \( \{ C_i | i \in I \} \) is any collection of closed right ideals of \( R \), then \( \bigcup_{i \in I} C_i = (\bigcap_{i \in I} C_i)^* \). \( (J^*_r ; \cap, \cup) \) will denote the lattice of all annihilator right ideals of \( R \). Then it is easily seen that \( J^*_r \subseteq L^*_r \). We note that the lattice \( J^*_r \) is not usually a sublattice of \( L^*_r \), although intersections are set-theoretic in both lattices. For convenience, we let \( L^*_r = L^*_r \cap L_2 \) and \( J^*_r = J^*_r \cap L_2 \), where \( L_2 \) is the set of two-sided ideals of \( R \). Corresponding left properties of a ring \( R \) are indicated by replacing each “\( r \)” by an “\( l \)”. If \( R \) is right locally uniform, then \( L^*_r \) is an atomic lattice, and \( A \in L^*_r \) is an atom if and only if \( A \) is a closed uniform right ideal. Following R. E. Johnson we call

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a ring $R$ a (right) potent ring (P-ring) if every nonzero closed right ideal of $R$ is potent. We say that uniform right ideals $A$ and $B$ are similar (in symbols; $A \sim B$) iff $A$ and $B$ contain mutually isomorphic nonzero right ideals $A'$ and $B'$ respectively. A ring $R$ said to be (right) irreducible iff $R$ is right locally uniform and $A \sim B$ for all uniform right ideals $A$ and $B$ of $R$. A right locally uniform irreducible ring with $Z_r(R) = 0$ is called here an I-ring. An I-ring which is also a P-ring will be called a PI-ring. We note that a ring $R$ is a PI-ring if and only if $R$ is a PI-ring in the sense of R. E. Johnson [6]. A ring $R$ is said to be residue-finite if the following condition is satisfied:

The factor ring $R/T$ is finite dimensional as a right $R$-module for any nonzero $T \in L^*_p$.

If $R$ is finite dimensional, then evidently $R$ is residue-finite. If $R$ is a prime ring, then $R$ is residue-finite, because $L^*_p = \{0, R\}$. A PI-ring which is countably dimensional will be called a CPI-ring. Let $M$ be a right $R$-module. If $M$ is an $n$-dimensional in the sense of Goldie, then we write $n = \dim_R M$.

Concerning the terminologies we refer to [4] and [6].

2. Residue-finite CPI-rings.

Theorem 1. If $R$ is a residue-finite CPI-ring, then the following properties hold:

1. $L^*_p = \bigcup_{A \in L^*_p : \text{atom}} A$.
2. $L^*_p$ is a chain and there exist the following two types:
   - (A): $R = T_0 \supset T_1 \supset T_2 \supset \cdots$ and $\bigcup_{p=0}^{\infty} T_p = 0$.
   - (B): There exists an integer $p$ such that $R = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_p \supset T_{p+1} = 0$.
3. For each nonzero $T_0 \in L^*_p$, there exists an independent set $\{A_1, \ldots, A_n\}$ of atoms of $L^*_p$ such that $A_1 \cup \cdots \cup A_n \cup T_p = T_{p-1}$ and $(A_1 \cup \cdots \cup A_n) \cap T_p = 0$.
4. If $A$ is an atom of $L^*_p$, then $A \subseteq T_p$ and $A \subseteq T_{p+1}$ if and only if $A' = T_{p+1}$.

The lattices $J^*_p$ and $J^*_p$ are dual isomorphic under the corresponding $A \rightarrow A^t, A \in J^*_p$. Hence if $J^*_p$ consists of $R = T_0 \supset T_1 \supset T_2 \supset \cdots, \bigcap_{p=0}^{\infty} T_p = 0$ or $R = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_p \supset T_{p+1} = 0$, then $J^*_p$ consists of $0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{p+1} = R$, respectively.

Lemma 1. Let $J^*_2 = \{T_0, T_1, T_2, \ldots\}$. Then:

1. For each $T_p \neq R$, there exists a potent atom $B \in J^*_p$ such that $B \subseteq T_{p+1}$ and $B \cap T_p = 0$.
2. If $B$ is a potent atom of $J^*_p$, then $B \subseteq T_{p+1}$ and $B \subseteq T_p$ if and only if $B' = T_p$.

By [5], the lattice $J^*_p$ is an upper semi-modular lattice. Hence for each $B \in J^*_p$ such that the interval $[0, B]$ is a finite length, we can define, by Theorem 14 of [1], the dimension of $B$ as the maximal length of chains.
between 0 and B. If the dimension of B is n, then we write \( n = \dim B \).

**Lemma 2.** (1) \( \dim_{\mathbb{R}}(R / T_p) = d_p \) if and only if \( \dim T_p = d_p \).

(2) For each nonzero \( T_p \), there exists an independent set \( \{B_1, \ldots, B_{d_p - 1 + 1}\} \) of potent atoms of \( J^* \) such that
\[
T_p = T_{p-1} \cup (B_{d_p - 1 + 1} \cup \cdots \cup B_{d_p}) \quad \text{and} \quad (B_{d_p - 1 + 1} \cup \cdots \cup B_{d_p}) \cap T_{p-1} = 0.
\]

Let \( \dim_{\mathbb{R}}(R / T_p) = d_p \) for each nonzero \( T_p \in \mathcal{L}_2 \). Then evidently \( \dim_{\mathbb{R}}(T_{p-1} / T_p) = d_p - d_{p-1} \). If \( R \) satisfies (A) in Theorem 1, we shall call the ring \( R \) of type (A), and \( (d_1, d_2 - d_1, \ldots, d_p - d_{p-1}, \ldots) \) is called a set of block numbers of \( R \).

If \( R \) satisfies (B) in Theorem 1, we shall call the ring \( R \) is of type (B), and \( (d_1, d_2 - d_1, \ldots, d_p - d_{p-1}, \infty) \) is called a set of block numbers of \( R \).

Let \( R \) be a ring with \( Z(R) = 0 \). As is well known the maximal right quotient ring \( \hat{R} \) of \( R \) is right \( R \)-injective and is a right self-injective (von Neumann) regular ring (see [2]). Let \( L \) be an atomic lattice with 1. A set \( \{a_i\} \) of atoms of \( L \) is independent iff \( a_i \cap (\bigcup_{j \neq i} a_j) = 0 \) for each \( i \). An independent set \( \{a_i\} \) of atoms of \( L \) is called a basis of \( L \) if \( \bigcup_i a_i = 1 \).

In order to make further progress we need the following definition:

Let \( R \) be a residue-finite PI-ring. \( R \) is said to be complemented with respect to \( \mathcal{L}_2 \) if there exists a set \( \{B_i\} \) of potent atoms of \( J^* \) such that

(a) \( \{B_i\} \) satisfies the condition (2) in Lemma 2, and

(b) For each nonzero \( T_p \), \( T_p \cup T_p^* = R \) and \( T_p \cap T_p^* = 0 \), where \( T_p^* = (\bigcup_{j > d_p} B_j)^* \). In addition, if \( \bigcup_p T_p^* = R \), then \( R \) is said to be s-complemented with respect to \( \mathcal{L}_2 \).

The following are examples of rings which are s-complemented with respect to \( \mathcal{L}_2 \):

(i) \( R \) is an FPI-ring in the sense of [6].

(ii) Let \( R \) be a residue-finite CPI-ring and let \( \hat{R} \) be the maximal right quotient ring of \( R \). If \( \hat{R} \) is a left quotient ring of \( R \), then \( R \) is s-complemented with respect to \( \mathcal{L}_2 \) (see [7]).

(iii) Let \( F \) be a division ring. If \( A \) and \( B \) are subsets of \( F \), then we denote by \( AB^{-1} \) the set \( \{ab^{-1} | a \in A, b \in B, b \neq 0\} \). Let \( \omega \) be the countable ordinal number. We denote by \( (F)_\omega \) the ring of all column-finite \( \omega \times \omega \) matrices over \( F \). Let \( F_{i,j} \) be additive subgroups of \( F \) such that \( F_{i,j} \subseteq F_{k,l} \) if \( i, j, k = 1, 2, \ldots \). Let \( S = \{a \in (F)_\omega | a = (a_{i,j}), a_{i,j} \in F_{i,j}\} \). Clearly \( S \) is a subring of \( R \). The ring \( S \) will be called a T-ring (triangular-block matrix ring) with type (A) in \( (F)_\omega \) iff there exist integers \( 0 = d_0 < d_1 < d_2 < \cdots < d_\eta < \cdots \) such that \( F_{i,j} \neq 0 \) iff \( i > d_p \) and \( d_p < j \leq d_{p+1}(p = 0, 1, 2, \ldots) \). If \( F_{i,j} \neq 0 \) = \( F \) and \( F_{i,j} F_{i,j}^* = F(2 \leq j < k) \), then \( S \) is s-complemented with respect to \( \mathcal{L}_2 \) and a residue-finite CPI-ring.
with type (A) (see [7], Theorem 2).

Let \( R \) be \( s \)-complemented with respect to \( L^*_2 \) with type (A) and let \( \{B_i\} \) be potent atoms of \( J^*_i \) which satisfies the conditions (a) and (b). Now we set \( A_i=\left(\bigcup_{j\neq i} B_j\right)^r \). Then the following lemma holds.

**Lemma 3.** (1) \( \{A_i\} \) and \( \{B_i\} \) are independent atoms of \( L^*_2 \) and \( J^*_i \) respectively.

(2) For each \( p \), \( T_{p^{-1}}=T_p \cup (A_{d_p+1} \cup \cdots \cup A_{d_p}) \) and \( T_p \cap (A_{d_p+1} \cup \cdots \cup A_{d_p})=\emptyset \).

(3) \( \bigcup A_i=R \).

(4) \( B_i=\left(\bigcup_{j\neq i} A_j\right)^i \).

Now, we can summarize the above-mentioned results as follows:

**Theorem 2.** Let \( R \) be a CPI-ring with type (A) and let \((d_1, d_2, \ldots , d_n, \cdots)\) be the set of block numbers of \( R \). If \( R \) is \( s \)-complemented with respect to \( L^*_2 \), then there exist potent atomic bases \( \{B_1, B_2, \ldots , B_n, \cdots\} \) for \( J^*_2 \) and \( \{A_1, A_2, \ldots , A_n, \cdots\} \) for \( L^*_2 \) such that:

(1) \( A_i=\left(\bigcup_{j\neq i} B_j\right)^r \) and \( B_i=\left(\bigcup_{j\neq i} A_j\right)^i \), \((i=1, 2, \cdots)\).

(2) \( J^*_2=L^*_2=\{A_i^{\geq i} \mid i=1, 2, \cdots\}, J^*_2=\{B_i^{\geq i} \mid i=1, 2, \cdots\} \).

(3) \( A_1^{\geq 1} \geq A_2^{\geq 2} \geq \cdots \geq A_i^{\geq i} \geq \cdots \), \( \bigcap_{n=1}^\infty A_n=\emptyset \) and \( 0=B_1^{\geq 1} \leq B_2^{\geq 2} \leq \cdots \leq B_n^{\geq n} \leq \cdots \), \( \bigcup_{n=1}^\infty B_n=R \).

(4) \( A_i^{\geq i}=A_j^{\geq j} \) and \( B_i^{\geq i}=B_j^{\geq j} \) iff \( d_0+d_1+\cdots+d_p<i \) and \( j\leq d_0+d_1+\cdots+d_{p+1} \) for some \( p \), where \( d_0=0 \).

(5) \( A_i B_j \neq 0 \) iff \( i>d_0+\cdots+d_p \) and \( d_0+\cdots+d_p<j\leq d_0+\cdots+d_{p+1} \) for some \( p \).

References


