226. *On Realization of the Discrete Series for Semisimple Lie Groups*

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This note is an announcement of a result, which says, briefly, that most of the discrete series for a semisimple Lie group are realized as certain eigenspaces of the Casimir operator on the symmetric space (Theorem 2). This construction is in some sense a generalization of the methods adopted in [1], [2], [9] for special groups and in [5] for the groups of hermitian type. Also, [6] indicates the above method of realization. Further, as for alternative methods to realize most of the discrete series, we refer to the recent works [5], [8]. Our technique used here depends heavily on that of [5]. A detailed exposition with full proofs will appear elsewhere.

1. Let G be a connected non-compact semisimple Lie group with a compact Cartan subgroup. We assume, for convenience, that G has a faithful finite dimensional representation and its complexification G^c is simply connected. Fix a maximal compact subgroup K of G and a Cartan subgroup H contained in K . We denote by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ the Lie algebras corresponding to G , K and H respectively. For complexifications $\mathfrak{g}^c$, $\mathfrak{k}^c$, $\mathfrak{h}^c$ of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, we denote by Δ the root system of $(\mathfrak{g}^c, \mathfrak{h}^c)$, and by W_G the Weyl group of $(\mathfrak{k}^c, \mathfrak{h}^c)$. Taking a positive root system P of Δ fixed once for all, P_k (resp. P_n) denotes the set of a positive compact (resp. non-compact) roots. Let L be the character group of H , L' the set of regular elements in L . Introducing an inner product $(\cdot)$ on L induced by the Killing form, we put $\varepsilon(\lambda) = \text{sign} \prod_{\alpha \in P} (\lambda, \alpha)$ for $\lambda \in L'$, and $\varepsilon(\lambda) = 0$ for $\lambda \in L - L'$. We also put $\varepsilon_k(\lambda) = \text{sign} \prod_{\alpha \in P_k} (\lambda, \alpha)$ if $\lambda \in L$ is $\mathfrak{k}^c$ -regular, and $\varepsilon_k(\lambda) = 0$ if λ is $\mathfrak{k}^c$ -singular. For discrete series, the following fact is known by Harish-Chandra [3]. Let $\mathcal{E}_d$ be the discrete series for G . For $\lambda \in L'$, there then exists a unique element $\omega(\lambda) \in \mathcal{E}_d$, and the map $L' \ni \lambda \mapsto \omega(\lambda) \in \mathcal{E}_d$ is surjective, while $\omega(\lambda) = \omega(\lambda')$ if and only if there exists $w \in W_G$ such that $w\lambda = \lambda'$. We shall denote by $\Theta_{\omega(\lambda)}$ the character of $\omega(\lambda)$.

For a finite subset A of L , we shall denote by $|A|$ its cardinal number and put $\langle A \rangle = \sum_{\alpha \in A} \alpha$. Put $\rho = \langle P \rangle / 2$, $\rho_k = \langle P_k \rangle / 2$ and $\rho_n = \rho - \rho_k$. If $\varepsilon_k(\lambda + \rho_k) \neq 0$ for $\lambda \in L$, there exists a unique $w \in W_G$ such that $w(\lambda + \rho_k) - \rho_k$ is k^c -dominant. We then denote by $[\lambda]$ the equivalence class to which belongs an irreducible K -module with highest weight

$w(\lambda + \rho_k) - \rho_k$. For the sake of notational convenience, we put $[\lambda] = 0$ if $\varepsilon_k(\lambda + \rho_k) = 0$. We shall denote by $\chi(\lambda)$ the character of $[\lambda]$.

2. For a finite dimensional unitary K -module V , we denote by $\mathcal{C}\mathcal{V}$ the homogeneous vector bundle over G/K associated to V , whose fiber has an invariant hermitian metric. Throughout this note, for a K -module the corresponding script letter denotes the homogeneous vector bundle associated to the K -module given. Let $L_2(\mathcal{C}\mathcal{V})$ (resp. $C^\infty(\mathcal{C}\mathcal{V})$) be a space of all square-integrable (resp. differentiable) sections of $\mathcal{C}\mathcal{V}$, which is naturally regarded as the space consisting of all V -valued square-integrable (resp. differentiable) functions f satisfying $f(gk) = k^{-1}f(g)$ for $k \in K, g \in G$. Now assume that there are given two K -modules V, W . For a G -invariant linear differential operator $D: C^\infty(\mathcal{C}\mathcal{V}) \rightarrow C^\infty(\mathcal{C}\mathcal{W})$, the maximal extension $D: L_2(\mathcal{C}\mathcal{V}) \rightarrow L_2(\mathcal{C}\mathcal{W})$ means the closed linear operator whose domain consists of $f \in L_2(\mathcal{C}\mathcal{V})$ such that $Df \in L_2(\mathcal{C}\mathcal{W})$ in the sense of distributions. We shall hereafter consider differential operators on square-integrable sections in this sense. Let $D^*: L_2(\mathcal{C}\mathcal{W}) \rightarrow L_2(\mathcal{C}\mathcal{V})$ be the maximal extension of the formal adjoint operator for D . We then have the unitary representations of G on the Hilbert spaces $\text{Ker } D$ and $\text{Ker } D^*$. Let $(\text{Ker } D)_d$ (resp. $(\text{Ker } D^*)_d$) be the smallest closed invariant subspace which contains every irreducible closed invariant subspace of $\text{Ker } D$ (resp. $\text{Ker } D^*$). Denote by π_V (resp. π_W) the representation on the space $(\text{Ker } D)_d$ (resp. $(\text{Ker } D^*)_d$). It is then shown that the operator $\pi_V(\varphi) = \int_G \varphi(g) \pi_V(g) dg$ is of trace class for a compactly supported C^∞ -function φ on G , and so defines an invariant distribution $\text{Trace } \pi_V$ on G (the same holds also for π_W). The following theorem can be proved by a similar method to the one in [5].

Theorem 1. *Under the above situation, assume that D is at most a first order operator, and denote by χ_V, χ_W the characters of V, W . Suppose that*

$$\chi_V - \chi_W = \varepsilon_k(\lambda + \rho) \sum_{Q \subset P_n} (-1)^{|Q_\lambda|} \chi(\lambda + \langle Q \rangle)$$

for some $\lambda \in L$ such that $\varepsilon(\lambda + \rho) \neq 0$. Then

$$\text{Trace } \pi_V - \text{Trace } \pi_W = (-1)^{|Q_\lambda|} \Theta_{\omega(\lambda + \rho)}$$

where $Q_\lambda = \{\beta \in P_n; (\lambda + \rho, \beta) > 0\}$.

Corollary. *For $\lambda \in L$, take such K -modules V, W as $[V] = \oplus [\lambda + \langle Q \rangle]$ where the summation runs over every $Q \subset P_n$ such that $\varepsilon_k(\lambda + \rho) \varepsilon_k(\lambda + \rho_k + \langle Q \rangle) = (-1)^{|Q|}$, and as $[W] = \oplus [\lambda + \langle Q \rangle]$ where the summation runs over every $Q \subset P_n$ such that $\varepsilon_k(\lambda + \rho) \varepsilon_k(\lambda + \rho_k + \langle Q \rangle) = (-1)^{|Q|+1}$. Then for any first order operator D , the formula in Theorem 1 holds. Here, $[V], [W]$ denote the equivalence classes to which V, W belong.*

3. Let $V_{\lambda + \langle Q \rangle}$ be an irreducible K -module belonging to $[\lambda + \langle Q \rangle]$ for $\lambda \in L, Q \subset P_n$, when $\varepsilon_k(\lambda + \rho_k + \langle Q \rangle) \neq 0$, and denote by $w_{\lambda + \langle Q \rangle}$ the

unique element of W_G such that $w_{\lambda+\langle Q \rangle}(\lambda + \rho_k + \langle Q \rangle)$ is k^c -dominant. Let Ω be the Casimir operator of G . Then the action of Ω on $L_2(G)$ as a left invariant differential operator defines the action $\nu(\Omega)$ on $L_2(\mathcal{CV}_{\lambda+\langle Q \rangle})$ because Ω belongs to the center of the universal enveloping algebra of $\mathfrak{g}^c$. Put $H_\lambda^Q = \{f \in L_2(\mathcal{CV}_{\lambda+\langle Q \rangle}) ; \nu(\Omega)f = (\lambda + 2\rho, \lambda)f\}$. For $w \in W_G$, we put $A_\lambda(w, Q) = (\rho - \langle Q \rangle, 2\langle Q_\lambda \rangle - \langle Q \rangle - \rho)/2 + (\rho_k, \rho_n - \langle Q_\lambda \rangle - w(\rho_n - \langle Q_\lambda \rangle)) + (\rho, \rho)/2$. We then have the following lemma in a similar way to the one in [6].

Lemma. *If $|(\lambda + \rho, \beta)| > A_\lambda(w_{\lambda+\langle Q \rangle}, Q)$ for every $\beta \in P_n$, then $H_\lambda^Q = 0$ if $Q \neq Q_\lambda$.*

In [4], we obtained an elliptic complex $\mathcal{CV}_\lambda^*$ whose first term is the homogeneous vector bundle associated to an irreducible K -module V_λ with lowest weight $\lambda + 2\rho_k$ (the $(\#)$ -complex for λ under an admissible linear order of Δ in terminology of [4]). One can define the square-integrable “cohomology” space $H_2^q(\mathcal{CV}_\lambda^*)$ for this elliptic complex. The following proposition is shown by Theorem 1 and the above Lemma.

Proposition. *There exists a non-negative constant a such that the following holds. If $|(\lambda + \rho, \alpha)| > a$ for every $\alpha \in P$, then $H_2^{q_\lambda}(\mathcal{CV}_\lambda^*) \neq 0$ and the irreducible unitary representation of G with character $\Theta_{\omega(\lambda+\rho)}$ is realized as a closed subspace of $H_2^{q_\lambda}(\mathcal{CV}_\lambda^*)$ for $q_\lambda = |Q_\lambda|$.*

4. For $\Delta \in L'$, choose a positive root system such as $P = \{\alpha \in \Delta ; (\Delta, \alpha) < 0\}$ and fix the linear order on Δ induced by P . Put $\lambda = \Delta - \rho$. Then $-(\lambda + 2\rho_k)$ is $\mathfrak{k}^c$ -dominant with respect to this linear order. Let V_λ be the irreducible K -module with lowest weight $\lambda + 2\rho_k$, and put $A(w, Q) = (\langle Q \rangle, \langle Q \rangle)/2 + (\rho_k, \rho_n - w\rho_n)$ and $b = \max_{w \in W_G, Q \subset P_n} A(w, Q)$. The next theorem follows from Corollary to Theorem 1 and Lemma in 3.

Theorem 2. *If $|(\lambda + \rho, \beta)| > b$ for every $\beta \in P_n$, then the Hilbert space*

$$\mathfrak{H}_\lambda = \{f \in L_2(\mathcal{CV}_\lambda) ; \nu(\Omega)f = (\lambda + 2\rho, \lambda)f\}$$

gives an irreducible unitary representation belonging to the discrete series for G , whose character is $\Theta_{\omega(\lambda+\rho)}$.

Remark. In view of Harish-Chandra's result cited in 1, we see that “most” of the discrete series are realized in this procedure. This construction is partially a generalization of the method in [9] for the de Sitter group and an answer to the proposal in [6]. Further, when (G, K) is a symmetric pair of hermitian type and that all elements in P_n are totally positive, Theorem 2 is included in Proposition 9.1 in [5].

5. As for a relation with another realization of the discrete series, we shall refer to the one by means of Schmid's operator (see [4], [7]). For $\Delta \in L'$, we choose P and define λ, V_λ as in 4. Put $c^\# = |\min_{\alpha \in P_k, Q \subset P_n} (\rho_n - \langle Q \rangle, \alpha)|$. Then λ satisfies the condition $(\#)$ in terminology of [4]

if $|(\lambda + \rho, \alpha)| \geq c^*$ for every $\alpha \in P_k$. Hence for a K -module V_λ^1 whose irreducible components consist of $[\lambda + \beta]$ where β runs over the elements on P_n , we have an elliptic first order operator

$$\mathcal{D}: L_2(\mathcal{CV}_\lambda) \rightarrow L_2(\mathcal{CV}_\lambda^1)$$

(see [4], [7]). We denote by $\mathcal{H}_\lambda$ the null space of $\mathcal{D}$. Put $c' = |\min_{\alpha \in P_k, \beta \in P_n} (\rho_k - \rho_n - \beta, \alpha)|$. It is then easily seen that the multiplicity of $[\lambda]$ in $\mathcal{H}_\lambda$ is at most one, if $|(\lambda + \rho, \alpha)| > \max(c^*, c')$ for every $\alpha \in P_k$, from Theorem 6.2 in [4] (see also [7]). Taking the unique element $w_0 \in W_G$ such that $w_0 P_k = -P_k$, we put $c'' = \max_{Q \subset P_n} A(w_0, Q)$, and $c = \max(c^*, c', c'')$. Combining the above fact with Theorem 1 and Lemma in 3, one can complete a proof of the following theorem.

Theorem 3*). *If $|(\lambda + \rho, \alpha)| > c$ for every $\alpha \in \Delta$, then $\mathcal{H}_\lambda$ gives an irreducible unitary representation with character $\Theta_{\omega(\lambda + \rho)}$.*

Remark. Under the condition of Theorem 3, we see that $\mathcal{H}_\lambda$ is contained in $\mathfrak{H}_\lambda$. Moreover, we can show that $\mathfrak{H}_\lambda$ is irreducible, which implies that $\mathfrak{H}_\lambda = \mathcal{H}_\lambda$. Therefore, under this condition, the two procedures to realize the discrete series are equivalent.

References

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*) The fact in Theorem 3 was communicated in the letter from Prof. Schmid without a proof.