28. Angular Cluster Sets and Horocyclic Angular Cluster Sets

By Hidenobu YOSHIDA

Department of Mathematics, Chiba University, Chiba

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1971)

1. In [1] Bagemihl began a study of relations between non-tangential (angular) boundary behaviors and horocyclic boundary behaviors of meromorphic functions defined in the open unit disk D of the complex plane. This study has been continued by Dragosh in [2] and [3]. The purpose of the present paper is to sharpen some of results of these investigations by the method of Dolzhenko's paper.

Notation and definitions. Unless otherwise stated, $f: D \rightarrow W$ shall mean $f(z)$ is an arbitrary function (generally not unique) defined in the open unit disk $D: |z| < 1$ and assuming values in the extended complex plane W . The unit circle $|z| = 1$ is denoted by Γ .

A circle internally tangent to Γ at a point $\zeta \in \Gamma$ is called a horocycle at ζ , and will be denoted by $h_r(\zeta)$, where r ($0 < r < 1$) is the radius of the horocycle.

Given a horocycle $h_r(\zeta)$ at a point $\zeta \in \Gamma$, the region interior to $h_r(\zeta)$ is called an oricycle at ζ , and will be denoted by $K_r(\zeta)$, or simply $K(\zeta)$ without specifying r . The half of $K_r(\zeta)$ lying to the right of the radius at ζ as viewed from the origin will be denoted by $K_r^+(\zeta)$ and $K_r^-(\zeta)$ denotes the left half of $K_r(\zeta)$ analogously.

Suppose that $0 < r_1 < r_2 < 1$. Let r_3 ($0 < r_3 < 1$) be so large that the circle $|z| = r_3$ intersects both of the horocycles $h_{r_1}(\zeta)$ and $h_{r_2}(\zeta)$. We define the right horocyclic angle $H_{r_1, r_2, r_3}^+(\zeta)$ at ζ with radii r_1, r_2, r_3 to be

$$H_{r_1, r_2, r_3}^+(\zeta) = \text{com}(\overline{K_{r_1}^+(\zeta)}) \cap K_{r_2}^+(\zeta) \cap \{z: |z| \geq r_3\},$$

where the bar denotes closure and com denotes complement, both relative to the plane. The corresponding left horocyclic angle is denoted $H_{r_1, r_2, r_3}^-(\zeta)$. We write $H_{r_1, r_2, r_3}(\zeta)$ to denote a horocyclic angle at ζ without specifying whether it be right or left, or simply $H(\zeta)$ in the event r_1, r_2, r_3 are arbitrary.

We assume the reader to be familiar with the rudiments of the cluster sets.

$C_V(f, \zeta)$, the angular cluster set of $f(z)$ at ζ on a Stolz angle $V(\zeta)$;

$C_K(f, \zeta)$, the oricyclic cluster set of $f(z)$ at ζ on an oricycle $K(\zeta)$;

$C_H(f, \zeta)$, the horocyclic angular cluster set of $f(z)$ at ζ on a horocyclic angle $H(\zeta)$.

A point $\zeta \in \Gamma$ is said to be a horocyclic angular Plessner point

(oricyclic Plessner point) of $f(z)$ provided that

$$C_H^+(f, \zeta) = W \text{ and } C_H^-(f, \zeta) = W \text{ } (C_K(f, \zeta) = W)$$

for each right and left horocyclic angle (each oricycle) at ζ .

A point $\zeta \in \Gamma$ is called a horocyclic angular Fatou point (oricyclic Fatou point) of $f(z)$ with a horocyclic angular Fatou value (an oricyclic Fatou value) $w \in W$ provided that

$$C_H^+(f, \zeta) = \{w\} \text{ and } C_H^-(f, \zeta) = \{w\} \text{ } (C_K(f, \zeta) = \{w\})$$

for each right and left horocyclic angle (each oricycle) at ζ .

Suppose a set $A \subset \Gamma$ and a point $\zeta = e^{i\theta} \in \Gamma$ are given. For a $\varepsilon > 0$, we denote an arc $\{e^{i\theta'}; \theta - \varepsilon < \theta' < \theta + \varepsilon\}$ by $\Gamma(\varepsilon, \zeta)$. Let $\gamma(\zeta, \varepsilon, A)$ be the largest of arcs contained in $\Gamma(\varepsilon, \zeta)$ and not intersecting with A . The set A is of porosity of the order α , $0 < \alpha \leq 1$ (or simply of porosity (α)) at ζ , if

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{\gamma(\zeta, \varepsilon, A)\}^\alpha > 0.$$

A is of porosity (α) on Γ if it is so at each $\zeta \in A$. A set which is a countable sum of sets of porosity (α) is said to be of σ -porosity (α).

A set of σ -porosity (α) is of the first Baire category.

It is easily seen that a measurable set which is of porosity (1) on Γ has no points of density. Therefore every measurable set of σ -porosity (1) on Γ is of measure 0. But there exists the set, which is of measure 0 and not of σ -porosity (1) (see [6], p. 75).

2. A $KH(KV)$ -singular point is the point $\zeta \in \Gamma$ such that $C_K(f, \zeta) \neq C_H(f, \zeta)$ ($C_K(f, \zeta) \neq C_V(f, \zeta)$) for some pair of $K(\zeta)$ and $H(\zeta)$ ($K(\zeta)$ and $V(\zeta)$). The set of all $KH(KV)$ -singular points is denoted by $E_{KH}(f)$ ($E_{KV}(f)$).

A KK -singular point is the point $\zeta \in \Gamma$ such that $C_{K'}(f, \zeta) \neq C_{K''}(f, \zeta)$ for some pair of oricycles $K'(\zeta)$ and $K''(\zeta)$. The set of all KK -singular points is denoted by $E_{KK}(f)$.

Let $\{r_i\}_{i=1}^\infty$ be a sequence of all rational numbers satisfying $0 < r_i < 1$, and let $\{D_n\}$ be a sequence consisting of all closed circles of the plane W having rational radii r_n and centers with rational coordinates.

For a $\varepsilon > 0$, we set $U_\varepsilon(\zeta) = \{z; |z - \zeta| < \varepsilon\}$. We denote $K_{r_p}(\zeta)$ by $\bar{K}_p(\zeta)$ and $H_{r_k, r_l, r_m}(\zeta)$ by $H_{k, l, m}(\zeta)$.

Lemma 1. *Let $\zeta \in A \subset \Gamma$. Suppose A is not of porosity (1) at a point $\zeta \in A$. Then for fixed $r_p, r_k, r_l, r_m, \bar{K}_p(\zeta) \cap U_\varepsilon(\zeta)$ is covered by the set $M = \bigcup_{\varepsilon \in A} H_{k, l, m}(\xi)$ supposed $\varepsilon > 0$ is sufficiently small.*

Proof. Without loss of generality, we may assume that $\zeta = 1$. Now we suppose that there exists a sequence $z_\nu = x_\nu + iy_\nu$ ($\nu = 1, 2, 3, \dots$) such that $z_\nu \in K_p(1) \cap U_\varepsilon(1) - M$ and $z_\nu \rightarrow 1$. For each z_ν , points $R_1(z_\nu)$, $S_1(z_\nu)$, $R_2(z_\nu)$, $S_2(z_\nu)$ on Γ are decided as follows.

$R_1(z_\nu)$ ($S_1(z_\nu)$) is the point on Γ such that the point z_ν lies on the right half of $h_{r_l}(R_1(z_\nu))$ ($h_{r_k}(S_1(z_\nu))$);

$R_2(z_\nu)(S_2(z_\nu))$ is the point on Γ such that the point z_ν lies on the left half of $h_{r_l}(R_2(z_\nu))(h_{r_k}(S_2(z_\nu)))$.

Let $z_\nu = r_\nu e^{i\theta_\nu}$. We immediately have

$$\begin{aligned} \overline{R_1(z_\nu)S_1(z_\nu)} & \text{ (the arc length connecting } R_1(z_\nu) \text{ and } S_1(z_\nu)) \\ &= \overline{R_2(z_\nu)S_2(z_\nu)} = \cos^{-1} \left\{ \frac{2(1-r_l)-(1-r_\nu^2)}{2(1-r_l)r_\nu} \right\} - \cos^{-1} \left\{ \frac{2(1-r_k)-(1-r_\nu^2)}{2(1-r_k)r_\nu} \right\}, \\ \overline{R_i(z_\nu)} \mathbf{1} &= \theta_\nu - (-1)^i \cos^{-1} \left\{ \frac{2(1-r_l)-(1-r_\nu^2)}{2(1-r_l)r_\nu} \right\} \quad (i=1, 2), \\ \overline{S_i(z_\nu)} \mathbf{1} &= \theta_\nu - (-1)^i \cos^{-1} \left\{ \frac{2(1-r_k)-(1-r_\nu^2)}{2(1-r_k)r_\nu} \right\} \quad (i=1, 2). \end{aligned}$$

Since $z_\nu \in K_p(1)$, $|\theta_\nu| < \cos^{-1} \left\{ \frac{2(1-r_p)-(1-r_\nu^2)}{2(1-r_p)r_\nu} \right\}$. We set

$$\begin{aligned} \varepsilon_\nu &= \max \{ \overline{R_1(z_\nu)} \mathbf{1}, \overline{S_1(z_\nu)} \mathbf{1}, \overline{R_2(z_\nu)} \mathbf{1}, \overline{S_2(z_\nu)} \mathbf{1} \}. \\ \lim_{\nu \rightarrow \infty} \frac{\overline{R_1(z_\nu)S_1(z_\nu)}}{\sqrt{1-r_\nu}} &> 0 \text{ and } \varepsilon_\nu = O(\sqrt{1-r_\nu}) \text{ as } \nu \rightarrow \infty. \end{aligned}$$

Since $\{R_1(z_\nu)S_1(z_\nu) \text{ (the arc connecting } R_1(z_\nu) \text{ and } S_1(z_\nu)) \cup R_2(z_\nu)S_2(z_\nu)\} \cap A = \phi$, we have $\gamma(1, \varepsilon_\nu, A) \geq \overline{R_1(z_\nu)S_1(z_\nu)}$.

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \gamma(1, \varepsilon, A) \geq \lim_{\nu \rightarrow \infty} \frac{1}{\varepsilon_\nu} \overline{R_1(z_\nu)S_1(z_\nu)} \geq \lim_{\nu \rightarrow \infty} \frac{\sqrt{1-r_\nu}}{\varepsilon_\nu} \frac{\overline{R_1(z_\nu)S_1(z_\nu)}}{\sqrt{1-r_\nu}} > 0,$$

and obtain a contradiction to the assumption that 1 is not a point of porosity (1) for A . Therefore, for $\varepsilon > 0$ small enough, $K_p(1) \cap U_\varepsilon(1)$ is covered by the set $M = \bigcup_{\varepsilon \in A} H_{k,l,m}(\hat{\xi})$.

Lemma 2 (Yanagihara [5, Theorem 1]). *Let $f: D \rightarrow W$. Then $E_{KK}(f)$ is of the type $G_{\delta\sigma}$ and σ -porosity (1).*

Theorem 1. *Let $f: D \rightarrow W$. Then $E_{KH}(f)$ is of the type $G_{\delta\sigma}$ and σ -porosity (1).*

Proof. $E_{n,k,l,m}$ is the set of points $\zeta \in \Gamma$ such that the set

$$\{w = f(z); z \in H_{k,l,m}(\zeta)\} \quad (1)$$

lies at a distance $\geq r_n$ from D_n .

$F_{n,p,q}$ is the set of points $\zeta \in \Gamma$ such that the set

$$\left\{ w = f(z); z \in K_p(\zeta), \frac{1}{3q} < \text{dis}(z, \zeta) < \frac{1}{q} \right\} \quad (2)$$

has common points with D_n .

Then $E_{n,k,l,m}$ is closed and $F_{n,p,q}$ is open. We put

$$F_{n,p} = \bigcap_{t=1}^{\infty} \bigcup_{q=t}^{\infty} F_{n,p,q} \text{ and } A_{n,k,l,m,p} = F_{n,p} \cap E_{n,k,l,m}. \quad (3)$$

We will show

$$E_{KH}(f) = \left(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \right) \cup E_{KK}(f). \quad (4)$$

Take a point $\zeta \in E_{KH}(f)$ and $\zeta \notin E_{KK}(f)$. There exist $K(\zeta)$ and $H(\zeta)$, $K(\zeta) \supset H(\zeta)$, for which $C_K(f, \zeta) \supseteq C_H(f, \zeta)$. Choose number p and s such that $K_p(\zeta) \supset K(\zeta)$ and

$$D_s \cap C_K(f, \zeta) \neq \phi, \quad \text{dis}(D_s, C_H(f, \zeta)) > 5r_s. \quad (5)$$

Then we can find numbers k, l, m such that $H(\zeta) \supset H_{k,l,m}(\zeta)$ and $\text{dis}(D_s, f(z)) > 4r_s$ for $z \in H_{k,l,m}(\zeta)$. If D_n is a disk with radius $r_n = 2r_s$ and concentric with D_s , $\text{dis}(D_n, f(z)) > r_n$ for $z \in H_{k,l,m}(\zeta)$, which shows $\zeta \in E_{n,k,l,m}$. In view of (5) there exists an infinite number of q such that

$$D_n \cap \left\{ w = f(z); z \in K_p(\zeta), \frac{1}{3q} < \text{dis}(z, \zeta) < \frac{1}{q} \right\} \neq \phi,$$

which shows $\zeta \in F_{n,p}$. Thus $\zeta \in A_{n,k,l,m,p}$ and

$$E_{KH}(\zeta) \subset \left(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \right) \cup E_{KK}(f).$$

Take a point $\zeta \in \left(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \right) \cup E_{KK}(f)$. If $\zeta \in E_{KK}(f)$, clearly we have $\zeta \in E_{KH}(f)$. If $\zeta \in \bigcup_{n,k,l,m,p} A_{n,k,l,m,p}$, $C_{H_{k,l,m}}(f, \zeta) \cap D_n = \phi$ from (1) and $C_{K_p}(f, \zeta) \cap D_n \neq \phi$ from (2). Thus we have $C_{H_{k,l,m}}(f, \zeta) \neq C_{K_p}(f, \zeta)$ and $\zeta \in E_{KH}(f)$. Hence $\left(\bigcup_{n,k,l,m,p} A_{n,k,l,m,p} \right) \cup E_{KK}(f) \subset E_{KH}(f)$.

Since $E_{KK}(f)$ is of type $G_{\delta\sigma}$ by Lemma 2, the equality (4) shows that $E_{KH}(f)$ is type $G_{\delta\sigma}$.

According to Lemma 1, $E_{KK}(f)$ is of σ -porosity (1), so that it remains to prove that $A = A_{n,k,l,m,p}$ is of porosity (1).

Suppose A is not of porosity (1) at a point $\zeta \in A$. Then for sufficiently small $\varepsilon > 0$, $K_p(\zeta) \cap U_\varepsilon(\zeta)$ is covered by the set $\bigcup_{\xi \in A} H_{k,l,m}(\xi)$ by Lemma 1. Thus if $z \in K_p(\zeta) \cap U_\varepsilon(\zeta)$, there is a point $\xi \in A = A_{n,k,l,m,p}$, $z \in H_{k,l,m}(\xi)$. Therefore $w = f(z)$ lies at a distance $\geq r_n$ from D_n , and $C_{K_p}(f, \zeta) \cap D_n = \phi$. This contradicts with $\zeta \in F_{n,p}$. Thus the porosity (1) of A is proved.

Theorem 2 (Yanagihara [5, Theorem 2]). *Let $f: D \rightarrow W$. Then $E_{KV}(f)$ is of the type $G_{\delta\sigma}$ and of σ -porosity (1/2).*

3. Now we can state some precisions and generalizations of the results of Bagemihl [1, Theorem 1, Theorem 2, Theorem 4 and Remark 3] and Dragosh [3, Theorem 1, Remark 2 and Corollary 2].

Theorem 3. *Let $f: D \rightarrow W$. Then a horocyclic angular Fatou point of $f(z)$ is an angular Fatou point of $f(z)$ except on a set of σ -porosity (1).*

Proof. According to Theorem 1, except on a set of σ -porosity (1), a horocyclic angular Fatou point of f is an oricyclic Fatou point of f , which is an angular Fatou point of f by the fact $C_K(f, \zeta) \supset C_V(f, \zeta)$.

Theorem 4. *Let $f: D \rightarrow W$. Then an angular Fatou point of $f(z)$ is a horocyclic angular Fatou point except on a set of σ -porosity (1/2).*

Proof. This is an analogous deduction from Theorem 2.

Theorem 5. *Let $f: D \rightarrow W$. Then an angular Plessner point of $f(z)$ is a horocyclic angular Plessner point of $f(z)$ except on a set of σ -porosity (1).*

Proof. By the fact $C_K(f, \zeta) \supset C_V(f, \zeta)$, an angular Plessner point

of f is an oricyclic Plessner point of f , which is a horocyclic angular Plessner point of f except on a set of σ -porosity (1) according to Theorem 1.

Theorem 6. *Let $f: D \rightarrow W$. Then a horocyclic angular Plessner point of $f(z)$ is an angular Plessner point of $f(z)$ except on a set of σ -porosity (1/2).*

Proof. This is an analogous deduction from Theorem 2.

4. We have not yet established the complete structural characterizations of set $E_i (i=1, 2, 3, 4)$ such that

$\left\{ \begin{array}{l} \text{a horocyclic angular Fatou point is an angular Fatou point on } E_1, \\ \text{an angular Fatou point is a horocyclic angular Fatou point on } E_2, \\ \text{an angular Plessner point is a horocyclic angular Plessner point on } E_3, \\ \text{a horocyclic angular Plessner point is an angular Plessner point on } E_4, \end{array} \right.$
 for some functions.

But we establish only one special result in this direction here

In proving next Theorem 7, we use the function $f(z)$ constructed by Yanagihara [5].

Lemma 3. *Let $E \subset \Gamma$ be a closed everywhere disconnected set. Then there exists a bounded holomorphic function $f(z)$ with the following properties:*

1) *At every $\zeta \in \text{comp}(F)$, $f(z)$ is continuous. Therefore, ζ is both an angular Fatou point and a horocyclic angular Fatou point.*

2) *Each point $\zeta \in F$ is an angular Fatou point having an angular limit of modulus 1.*

3) *Each point $\zeta \in F$ at which the set F is of porosity (1/2) is not a horocyclic angular Fatou point.*

Proof. $\text{Comp}(F)$ consists of a countable number of arcs $(\zeta'_\nu, \zeta''_\nu)$. For a constant $r (0 < r < 1)$, $h_r(\zeta'_\nu) \cap h_r(\zeta''_\nu) \neq \emptyset$ except at most finite number of ν 's. Let $z'_{\nu,1} = z''_{\nu,1}$ be the one of intersection points of $h_r(\zeta'_\nu)$ and $h_r(\zeta''_\nu)$ which is nearer to Γ . For each exceptional index ν , let $z'_{\nu,1}$ be the left one of intersection points of $h_r(\zeta'_\nu)$ and $|z|=1-r$, and let $z''_{\nu,1}$ be the right one of intersection points of $h_r(\zeta''_\nu)$ and $|z|=1-r$. Next, let $z'_{\nu,n}$ be the point on $h_r(\zeta'_\nu)$ such that

$$\frac{1-|z'_{\nu,n}|}{|\zeta'_\nu - z'_{\nu,n}|} = \frac{1}{2} \frac{1-|z'_{\nu,n-1}|}{|\zeta'_\nu - z'_{\nu,n-1}|} \quad (n=2, 3, 4, \dots).$$

The sequence $\{z''_{\nu,n}\}$ on $h_r(\zeta''_\nu)$ is defined analogously.

Then the Blaschke product

$$f(z) = \prod \frac{\overline{z'_{\nu,n}}}{|z'_{\nu,n}|} \frac{z - z'_{\nu,n}}{1 - \overline{z'_{\nu,n}}z} \prod \frac{\overline{z''_{\nu,n}}}{|z''_{\nu,n}|} \frac{z - z''_{\nu,n}}{1 - \overline{z''_{\nu,n}}z}$$

has the properties asserted in Lemma 3.

Now, we shall prove here the property 3) only.

If we choose appropriate constants r_1, r_2, r_3 and $\zeta \in F$ is a point at

which the set F is of porosity $(1/2)$, $H_{r_1, r_2, r_3}(\zeta)$ contains an infinite number of points from $\{z'_{\nu, n}, z''_{\nu, n}\}$. Hence, for each point ζ at which the set F is of porosity $(1/2)$, if ζ is a horocyclic Fatou point of $f(z)$, then the horocyclic Fatou value at ζ must be 0, so that by the theorem of Lindelöf, the angular Fatou value at ζ must be 0, too. But this contradicts the property 2).

Theorem 7. *For each set of σ -porosity $(1/2)$ $E \subset \Gamma$, there exists a bounded analytic function $f(z)$ for which each $\zeta \in E$ is an angular Fatou point and is not a horocyclic angular Fatou point.*

Proof. E can be represented in the form of a countable sum of sets E_n nowhere dense on Γ : $E = \bigcup E_n$. Suppose that a set E' is a countable sum of closed everywhere disconnected sets $\overline{E}_n$: $E' = \bigcup \overline{E}_n$. Then E' can also be represented in the form of a not more than countable sum of closed sets F_k without common points (Dolzhenko [4], English translation, p. 8).

From this construction, it is evident that each point $\zeta \in \overline{E}_n$ at which some E_n is of porosity $(1/2)$ is also a point of porosity $(1/2)$ for some F_k .

Now, for each F_k we construct a sequence of zeros $\{z^k_{\nu, n}\}$ and Blaschke product $f_k(z)$, as in Lemma 3. Set

$$f(z) = \sum 2^{-k} f_k(z).$$

If $\zeta \in E_k$, all $f_{\nu}(z) (\nu \neq k)$ are continuous at ζ (Lemma 3, 1)), and $f_k(z)$ has an angular limit of modulus 1 (Lemma 3, 2)). Hence each point $\zeta \in E$ is an angular Fatou point. On the other hand, if $\zeta \in E_n$ is a point of porosity $(1/2)$ for F_k , then ζ is not a horocyclic angular Fatou point of $f_k(z)$ (Lemma 3, 3)). Thus each point $\zeta \in E$ is not a horocyclic angular Fatou point of $f(z)$.

Theorem 7 corresponds to Theorem 4. In this connection, it is natural to ask the following question: Is it true that there hold analogous results to Theorem 7 corresponding to other theorems in section 3? I would guess that it is positive even for holomorphic functions.

References

- [1] Bagemihl, F.: Horocyclic boundary properties of meromorphic functions. Ann. Acad. Sci. Fenn., A I, **385**, 1–18 (1966).
- [2] Dragosh, S.: Horocyclic boundary behavior of meromorphic functions. J. d'Anal. Math., **22**, 37–48 (1969).
- [3] —: Horocyclic cluster sets of functions defined in the unit disk. Nagoya Math. J., **35**, 53–82 (1969).
- [4] Dolzhenko, E. P.: Boundary properties of arbitrary functions. Izvestija, Acad. Nauk SSSR, **31**, 3–14 (1967).
English translation: Math. of the USSR-IZVESTIJA, **1**, 1–12 (1967).
- [5] Yanagihara, N.: Angular cluster sets and oricyclic cluster sets. Proc. Japan Acad., **45**, 423–428 (1969).
- [6] Collingwood, E. F., and Lohwater, A. J.: The Theory of Cluster Sets. Camb. Univ. Press (1966).