§ 1. Introduction. We have discussed in [2] the hypoellipticity of linear partial differential operators of the form
\[ P = \frac{\partial}{\partial t} + L(t, x; D_x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \]
where \( D_x = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n) \) and \( L(t, x; \xi) \) is a polynomial in \( \xi \in \mathbb{R}^n \) of order \( 2\mu \) with coefficients in \( C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x) \). In particular we have been interested in operators which are called to be of Fokker-Plank type. These were transformed by a change of independent variable into one having properties (O), (I), (II) and (III) stated in Proposition 1 and Remark of [2] (see also Theorem 3 in § 2), and we could show that if an operator possesses these properties, it has a very regular right-parametrix (see Theorem 3 in § 2) and hence its transpose is hypoelliptic. Applying this theorem with \( I = [-1, 1] \) and \( D = \{(t, s); -1 \leq s < t \leq 1\} \), we can prove, for example, the following

**Theorem 1.** Let, for real \( r, \langle r \rangle \) be an integer such that \( r \leq \langle r \rangle < r + 1 \) and \( M_j(t, x; \xi) \) a polynomial in \( \xi \in \mathbb{R}^n \) of homogeneous order \( j \) with coefficients in \( C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x) \). Then both the operator
\[ P = \frac{\partial}{\partial t} + \sum_{j=0}^{2\langle r \rangle} M_j(t, x; D_x), \quad l = 0, 1, \ldots, \]
and its transpose \( tP \) are hypoelliptic in \( \mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x \), if \( l \) is even and if for every compact set \( K \) of \( \mathbb{R}^{n+1} \) there exists a constant \( \delta > 0 \) such that
\[ \text{Re} \, M_{2\langle r \rangle}(t, x; \xi) \geq \delta |\xi|^2_n, \quad (t, x) \in K, \xi \in \mathbb{R}^n. \]
For the proof we use (9) with \( t \in [-1, 1] \) and \( (t, s), -1 \leq s < t \leq 1 \), and Lemmas 1 and 2 in § 4.

On the other hand Kannai proved recently in [1] that the operator
\[ \frac{\partial}{\partial x} - xD_y, \quad D_y = -i\frac{\partial}{\partial y} \]
is hypoelliptic in the plane and moreover its transpose
\[ -i\frac{\partial}{\partial x} - xD_y \]
is not locally solvable there, of course not hypoelliptic. As an extension of this result we can give

**Theorem 2.** The transpose of operator (2), \( tP \), with odd \( l \) is
hypoelliptic in \( R^{n+1} \), if condition (3) is satisfied for every compact set \( K \) of \( R^{n+1} \). Moreover, in case the coefficients of \( P \) are independent of \( x \), \( P \) is not hypoelliptic there.

This is a corollary of Theorem 4 which is stated in § 2 and whose proof will be completed in § 3 by using Theorem 3 in § 2 and the reasoning adapted in [1]. The proof of Theorem 2 will be briefly given in § 4.

§ 2. Statement of the main theorems. The following theorem is an amelioration of one given in [2].

**Theorem 3.** Suppose that the \( L(t, x ; D_x) \) in operator \( P \) of the form (1) possesses real \( n \)-square matrices \( \Gamma \) and \( T(t, s) \) with entries in \( C^0(I) \) and \( C^0(\Delta) \), respectively, which have the following properties (\( I = [0, 1] \) and \( \Delta = \{(t, s); 0 < s < t \leq 1\} \)):

1. There exists a constant \( \nu > 0 \) such that \( (t-s)^\nu \| T(t, s) \|^\nu \) is bounded in \( \Delta \).
2. If \( L^0(t, x ; \xi) \) denotes the leading part of \( L(t, x ; \xi) \), then for every compact domain \( \Omega \) of \( R^n \) there exists a positive constant \( \delta \) such that, for every \( (t, s) \in \Delta \) and \( x \in \Omega \),
   \[
   \text{Re} \int_s^t L_0(\tau, x ; T(t, s)\xi) d\tau \geq \delta |\xi|^p, \quad \xi \in R^n.
   \]
3. Let \( \Omega \) be an arbitrary compact domain of \( R^n \). Then the coefficients of the polynomial in \( \xi \),
   \[
   \int_s^t L(\tau, x ; T(t, s)\xi) d\tau,
   \]
   are all bounded in \( \Delta \times \Omega \).

4. The \( L(t, x ; \xi) \) is written as a polynomial of \( \Gamma, \xi \) with coefficients in \( C^0(I)(C(R^n)) \) and the inequality
   \[
   |\Gamma, \xi| \leq \text{const.} |\Gamma(t, s)\xi|^p, \quad \xi \in R^n,
   \]
   is valid for every \( (t, s) \in \Delta \), if we put
   \[
   \Gamma(t, s) = (t-s)^{-1/p} T(t, s)^{-1}.
   \]

Then, for each \( x_0 \in R^n \), there exist an open neighborhood \( V \) of \( x_0 \) and two sequences of distributions on \( W = (-1, 1) \times V \),

\[
\{E^{(p)}(t, x ; s, y)\}, \{R^{(p)}(t, x ; s, y)\} \quad (p=1, 2, \cdots),
\]

such that \( E^{(p)} = 0 \) and \( R^{(p)} = 0 \) for every \( p \) and for \( t < s \), satisfying the following, for every \( p \),

1. \( P(t, x) E^{(p)} = \delta(t-s) \times \delta(x-y) - R^{(p)} \),
2. \( E^{(p)} \in C^\infty(W - \{(t, x ; s, y); (t, x) = (s, y)\}) \),
3. for every \( \varphi(s, y) \in C^\infty((0, 1) \times V) \),
   \[
   \left< E^{(p)}, \varphi \right>_{(s, y)} \in C^\infty((-1, 1) \times V),
   \]
4. for every \( \psi(t, x) \in C^\infty((-1, 1) \times V) \)

1) By \( \| T \| \) we denote supremum of the set \( \{T \xi; |\xi| = 1\} \).
2) \( a(t, x) \in C^0(I)(C(R^n)) \) means that the mapping \( t \rightarrow a(t, x) \in C(R^n) \) is continuous in \( I \).
3) We wrote in [2] as \( \| \Gamma, \xi \| \leq \text{const.} \| \Gamma(t, s) \xi \|_p \), but it is not sufficient.
\[ \langle E^{(p)}, \psi \rangle_{(t, x)} \in C_0^\infty((0, 1) \times V), \]
(P. 5) for any integer \( N > 0 \), there exists an integer \( p_0 > 0 \) such that
\[ R^{(p)} \in C^N(W) \quad \text{for all} \quad p \geq p_0. \]

Their two sequences of distributions on \( W \), \( \{E^{(p)}\} \) and \( \{R^{(p)}\} \), are called a very regular right-parametrix in \( W \) of \( P \). The proof of Theorem 3 has been essentially established in [2]. We would make an additional remark that the property (III) can be dropped in case the coefficients of \( L \) are independent of \( x \).

Before ending this section we state the main theorem in this note:

**Theorem 4.** Suppose that the \( L(t, x ; D_x) \) in operator \( P \) of the form (1) and \( -L(-t, x ; D_x) \) both satisfy the hypothesis of Theorem 3. Then \( tP \) is hypoelliptic in \( R^{n+1} \).

This will be proved in the next section

\[ \S \ 3. \ \text{Proof of Theorem 4.} \] We give in this section the proof of Theorem 4. Throughout this section we denote by \( P \) an operator satisfying the condition mentioned in Theorem 4. It has been established in [2] that \( tP \) is hypoelliptic in \((R - \{0\}) \times R^n\). Therefore, for the proof of hypoellipticity of \( tP \) in \( R^{n+1} \), it suffices to show that \( tP \) is hypoelliptic in \((-1, 1) \times R^n\).

First, it follows from Theorem 3 that \( P \) has a very regular right-parametrix in \( W \) satisfying (P. 1)~(P. 5), since \( L(t, x ; D_x) \) satisfies the hypothesis in Theorem 3. Let \( V \) be an open set stated in Theorem 3, \( G = (-1, 1) \times V \) and \( u \) be a distribution on \( G \) satisfying \( tPu \in C^\infty(G) \).

Taking two domains \( G_1, G_2 \) and a function \( \beta \in C_0^\infty(G) \) so that \( G_1 \subset G_1 \subset G_2 \subset G \) and \( \beta = 1 \) on \( G_2 \), we have
\[ tP(\beta u) = \beta tPu + X, \]
where \( \beta tPu \) is in \( C_0^\infty(G) \), and \( X \) is a distribution on \( G \) with compact support and vanishes on \( G \). It then follows from (P. 2) and (P. 4) that
\[ \langle E^{(p)}(t, x ; s, y), tP(\beta u) \rangle_{(t, x)} \in C^\infty(G_1 \cap ((0, 1) \times V)). \]

By (P. 1) and (P. 3) we have
\[ (\beta u)(s, y) = \langle E^{(p)}, tP(\beta u) \rangle_{(t, x)} + \langle R^{(p)}, \beta u \rangle_{(t, x)} \]
for all \( p \) and \( s > 0 \). On the other hand we can assert by (P. 5) that for any integer \( N > 0 \), there exists an integer \( p_1 > 0 \) such that
\[ \langle R^{(p)}, \beta u \rangle_{(t, x)} \in C^N([0, 1) \times V) \]
for all \( p \geq p_1 \). Thus we finally obtain by (4) that the right hand side of (5) is in \( C^\infty(G_1 \cap ((0, 1) \times V)) \) for all \( p \geq p_1 \). So that \( u \) is infinitely differentiable in \( G_1 \cap ((0, 1) \times V) \) and hence \( u \) is in \( C^\infty([0, 1) \times V) \). It follows similarly from the assumption on \(-L(-t, x ; D_x)\) that \( u \) is also in \( C^\infty((-1, 0]) \times V) \).

By the same argument as in [1] we can see that \( u \) is in \( C^\infty(G) \). In fact, let \( \bar{u} \) be a distribution on \( G \) defined by
\[ \langle \bar{u}, \varphi \rangle = \left( \int_0^1 \int_{-1}^1 \int_{-1}^1 \right) u(t, x) \varphi(t, x) \, dt \, dx, \quad \varphi \in C^0_0(G). \]

Set \( v = u - \bar{u} \). Obviously supp \( [v] \) is on the hyperplane \( t = 0 \). Therefore, denoting by \( V_0 \) a compact subdomain of \( V \), we can find a finite number of distributions on \( V_0 \), \( v_j \) (\( j = 1, \ldots, N \)), such that

\[ v = \sum_{j=0}^N (Ev_j) \left( \frac{\partial}{\partial t} \right)^j \text{ on } (-1, 1) \times V_0, \]

where \( Ev_j \) are distributions on \((-1, 1) \times V_0\) defined by

\[ \langle Ev_j, \varphi(t, x) \rangle = \langle v_j, \varphi(0, x) \rangle, \quad \varphi \in C^0_0((-1, 1) \times V_0). \]

Calculating we obtain

\[ \langle Pu, \varphi \rangle = \langle Pu - E[u(+0, x) - u(-0, x)] , \varphi \rangle \]

for \( \varphi \in C^0_0((-1, 1) \times V_0) \). Hence

\[ \langle Pu, \varphi \rangle = \langle Pu - E[u(+0, x) - u(-0, x)] , \varphi \rangle \]

Thus it follows from (6), (7) and (8) that \( v = 0 \) and hence \( \mu = \bar{u} \). Therefore, by (8) we have \( u(+0, x) = u(-0, x) \). Consequently \( u \in C^0(G) \).

Now, taking account of the fact that \( \frac{\partial u}{\partial t} \in C^0(G) \), we can assert, by the same argument as above, \( \frac{\partial u}{\partial t}(+0, x) = \frac{\partial u}{\partial t}(-0, x) \) and so on.

This completes the proof of Theorem 4.

\( \$ 4. \) Proof of Theorem 2. We are going to prove Theorem 2. It is assumed that \( P \) is written in the form (2) with odd \( l \) and satisfies the hypothesis in Theorem 2. For the proof, we have only to show that

\[ L(t, x ; D_x) = \sum_{j=0}^{2p} t^{j/(2p)} M_j(t, x ; D_x) \]

and \( -L(-t, x ; D_x) \) satisfy the hypothesis of Theorem 3. To do so, we have only to choose

\[ T(t, \alpha) = \left( \frac{l+1}{t^{l+1}} \right)^{1/2p} \alpha \]

where \( I_n \) is the identity matrix of order \( n \). In fact these matrices have the properties (O), (I), (II) and (III) in Theorem 3. This can be verified by using the following two lemmas.

**Lemma 1.** Let \( \alpha \) be real and \( \alpha \geq 1 \). Then we have

\[ \frac{(x-y)\alpha}{x^\alpha - y^\alpha} \leq 1 \quad \text{for } 0 \leq y < x. \]

**Lemma 2.** For any integer \( l \geq 0 \), there exists a constant \( C_l > 0 \) such that

\[ s^l \leq C_l \frac{t^{l+1} - s^{l+1}}{t-s} \]

for \( t > s \) in case \( l \) is even and for \( t > s \geq 0 \) in case \( l \) is odd.
Thus it follows from Theorem 4 that $iP$ is hypoelliptic in $R^{n+1}$.

The latter half of Theorem 2 is showed as follows. Let the coefficients of the $P$ be independent of $x$. For every $\xi \in R^n$, we introduce, as in [1], functions $u_\xi(t, x)$ defined in $(-1, 1) \times R^n$ as

$$u_\xi(t, x) = \exp \left\{ i x \xi - \sum_{j=0}^{2n} \int_0^t \tau^{(j+1)/2} M_j(\tau; \xi) d\tau \right\}.$$ 

Obviously, these are solutions of $Pu = 0$. It now follows from (I), (II) and the $T_{(t, x)}$ in (9) that there exist positive constants $c$ and $C$ such that

$$|u_\xi(t, x)| \leq C \exp \{-c |T_{(t, 0)}^{-1} \xi|^{2n}\}, \quad \xi \in R^n,$n

for every $(t, x) \in (-1, 1) \times R^n$. Thus, if we take a real number $s$ so that $s > 2\mu + n$, the function determined by

$$u(t, x) = \int (1 + |\xi|^{-s}) u_\xi(t, x) d\xi$$

satisfies the equation $Pu = 0$ but is not infinitely differentiable in $(-1, 1) \times R^n$. This shows that $P$ is not hypoelliptic in $R^{n+1}$.

References