178. On Countably R -closed Spaces. I

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A topological space S is called *countably R -closed*, if for any family $\{G_n\}_{n=1}^{\infty}$ of nonvoid open sets such that $G_n \supset \bar{G}_{n+1}$ for every n , we have $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. Z. Frolik [1] proved the following:

Proposition. *In any topological space S , the following properties are equivalent:*

- (i) S is countably R closed.
- (ii) Every star-finite open covering of S has a finite subfamily whose union is dense in S .
- (iii) Every star-finite open covering of S has a finite subcovering.
- (iv) Every star-finite open covering of S is a finite covering.

We shall give other characterizations of countably R -closed spaces. In a topological space S , a family Φ composed of subsets of S is called *locally finite (discrete)* if every point x has a neighbourhood $U(x)$ which meets only finite members (at most only one member) of Φ , and Φ is called *star-finite* if every member of Φ meets only finite members of Φ . A subset E is called *regularly closed* if E is the closure of an open set of S . A covering of S composed of regularly closed sets is called a *regularly closed covering* of S .

Theorem. *In any topological space S , the following conditions are equivalent:*

- (1) S is countably R -closed.
- (2) Every locally finite, star-finite, countable, regularly closed covering of S has a finite subcovering.
- (3) Every locally finite, star-finite, countable, regularly closed covering of S is a finite covering.
- (4) Every locally finite, star-finite, regularly closed covering of S is a finite covering.
- (5) Every star-finite open covering of S is a finite covering.

We shall prove that (1) $\rightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (1) and (3) $\rightarrow$ (4) $\rightarrow$ (5) $\rightarrow$ (4) $\rightarrow$ (3).

Lemma 1. *In a topological space S , let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a locally finite, countable, regularly closed covering of S . Then $F_n = \bigcup_{k=n+1}^{\infty} \bar{O}_k$ is closed, $G_n = S - \bigcup_{i=1}^n \bar{O}_i$ is open, and $F_n \supset G_n$ for every n .*

Lemma 2. *Let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a locally finite, star-finite, countable, regularly closed covering of S . For every n , there is m ($\geq n+1$) such that $F_m \subset G_n$.*

Proof that (1)→(2). Let S be a topological space and let $\{\bar{O}_n\}_{n=1}^\infty$ be a locally finite, star-finite, countable, regularly closed covering of S . Suppose that $\{\bar{O}_n\}_{n=1}^\infty$ has no finite subcovering of S . Then $S - \bigcup_{l=1}^n \bar{O}_l \neq \phi$ i.e. $G_n \neq \phi$ for every n . Put $n_1=1, F_{n_1}=\bigcup_{k=2}^\infty \bar{O}_k$ and $G_{n_1}=S-\bar{O}_1$. By Lemma 1, F_{n_1} is closed, G_{n_1} is open, and $F_{n_1} \supset G_{n_1}$. Suppose that a sequence $\{n_t\}_{t=1}^s$ has been defined such that $n_1 < n_2 < \dots < n_s, F_{n_t} (1 \leq t \leq s)$ are closed, $G_{n_t} (1 \leq t \leq s)$ are nonvoid open, and $F_{n_1} \supset G_{n_1} \supset F_{n_2} \supset G_{n_2} \supset \dots \supset F_{n_s} \supset G_{n_s}$. By Lemma 2, for n_s there is $m (\geq n_s + 1)$ such that $G_{n_s} \supset F_m \supset G_m$ where F_m is closed and G_m is nonvoid open. Put $n_{s+1}=m$. Thus we have a sequence $\{n_t\}_{t=1}^{s+1}$ such that $n_1 < n_2 < \dots < n_s < n_{s+1}, F_{n_t} (1 \leq t \leq s+1)$ are closed, $G_{n_t} (1 \leq t \leq s+1)$ are nonvoid open, and $F_{n_1} \supset G_{n_1} \supset \dots \supset F_{n_{s+1}} \supset G_{n_{s+1}}$. Then there is a sequence $\{n_t\}_{t=1}^\infty$ such that $n_1 < n_2 < \dots < n_t < \dots, F_{n_t} (1 \leq t < \infty)$ are closed, $G_{n_t} (1 \leq t < \infty)$ are nonvoid open, and $F_{n_1} \supset G_{n_1} \supset \dots \supset F_{n_t} \supset G_{n_t} \supset \dots$ i.e. $G_{n_1} \supset \bar{G}_{n_2} \supset G_{n_2} \supset \dots \supset G_{n_t} \supset \bar{G}_{n_{t+1}} \supset \dots$. If the space S were countably R -closed, we should have $\bigcap_{t=1}^\infty G_{n_t} \neq \phi$. Then, there should exist a point x_0 such that $x_0 \in G_{n_t}$ i.e. $x_0 \in \bigcup_{l=1}^{n_t} \bar{O}_l$ for every t . Then $x_0 \in \bigcup_{l=1}^\infty \bar{O}_l$, contrary to that $\{\bar{O}_n\}_{n=1}^\infty$ is a covering of S . Therefore S may not be countably R -closed.

Proof that (2)→(3). Let S be a topological space and let us assume that any locally finite, star-finite, countable, regularly closed covering $\{\bar{H}_n\}_{n=1}^\infty$ of S , has a finite subcovering $\{\bar{H}_{n_k}\}_{k=1}^m$. If infinitely many different $\bar{H}_n$ were not nonvoid, there should exist a member $\bar{H}_{n_k}$ which met infinitely many different nonvoid $\bar{H}_n$. This is contrary to the condition that $\{\bar{H}_n\}_{n=1}^\infty$ is star-finite. Then the family $\{\bar{H}_n\}_{n=1}^\infty$ must be finite.

Lemma 3. In any topological space S , if there is a family $\{G_n\}_{n=1}^\infty$ of open sets where $G_n \supset \bar{G}_{n+1}$ for every n and $\bigcap_{n=1}^\infty G_n = \phi$, the family $\{\bar{H}_n\}_{n=0}^\infty$, where $H_0 = S - \bar{G}_1$ and $H_n = G_n - \bar{G}_{n+1}$ for every $n (\geq 1)$, is a locally finite, star-finite, regularly closed covering of S .

Proof. Put $K_0 = S - \bar{G}_3, K_n = G_n - \bar{G}_{n+3}$ for every $n (\geq 1)$. In the first place, we shall prove that $\bar{H}_0 \subset K_0, \bar{H}_n \subset K_{n-1}$ for every $n (\geq 1)$. Since $H_0 = S - \bar{G}_1 \subset S - G_1$ and $S - G_1$ is closed, we have $\bar{H}_0 \subset S - G_1 \subset S - \bar{G}_3 = K_0$. Since $H_n = G_n - \bar{G}_{n+1} \subset \bar{G}_n - G_{n+1}$ and $\bar{G}_n - G_{n+1}$ is closed for every $n (\geq 1)$, we have $\bar{H}_n \subset \bar{G}_n - G_{n+1} \subset G_{n-1} - \bar{G}_{n+2} = K_{n-1}$, assuming that $G_0 = S$. Since the family $\{K_n\}_{n=0}^\infty$ of open sets is star-finite covering of S and $\bar{H}_0 \subset K_0, \bar{H}_n \subset K_{n-1}$ for every $n (\geq 1)$, we have that the family $\{\bar{H}_n\}_{n=0}^\infty$ is locally finite and star-finite. Next we shall prove that $\{\bar{H}_n\}_{n=0}^\infty$ is a covering of the space S . Since the family $\{K_n\}_{n=0}^\infty$ is a covering of S and $K_n = G_n - \bar{G}_{n+3} = H_n \cup (\bar{G}_{n+1} - G_{n+1}) \cup H_{n+1} \cup (\bar{G}_{n+2} - G_{n+2}) \cup H_{n+2}$ for every $n (\geq 0)$, it is sufficient to prove that $\bar{G}_n - G_n \subset \bar{H}_n$ for every $n (\geq 1)$. If $\bar{G}_n - G_n = \phi$, it is trivial. If $\bar{G}_n - G_n \neq \phi$, let x be an arbitrary point of $\bar{G}_n - G_n$. Since $x \in \bar{G}_n - G_n \subset G_{n-1} - \bar{G}_{n+1}$ and $G_{n-1} - \bar{G}_{n+1}$ is open, $G_{n-1} - \bar{G}_{n+1}$ is a neighbourhood $V_1(x)$ of x . Since $x \in \bar{G}_n - G_n \subset \bar{G}_n$, for

any neighbourhood $V(x)$ of x , we have $V(x) \cap V_1(x) \cap G_n \neq \phi$. Hence, for any neighbourhood $V(x)$ of x ,

$$\begin{aligned} V(x) \cap H_n &= V(x) \cap (G_n - \bar{G}_{n+1}) = V(x) \cap (G_{n-1} - \bar{G}_{n+1}) \cap G_n \\ &= V(x) \cap V_1(x) \cap G_n \neq \phi. \end{aligned}$$

Then $x \in \bar{H}_n$ i.e. $\bar{G}_n - G_n \subset \bar{H}_n$. Thus we have proved that $\{\bar{H}_n\}_{n=0}^\infty$ is a covering of the space S .

Proof that (3) $\rightarrow$ (1). Let us assume that in a topological space S any locally finite, star-finite, countable, regularly closed covering $\{\bar{O}_n\}_{n=1}^\infty$ of S is a finite covering. If there is a family $\{G_n\}_{n=1}^\infty$ of open sets where $G_n \supset \bar{G}_{n+1}$ for every n and $\bigcap_{n=1}^\infty G_n = \phi$, we put $H_0 = S - \bar{G}_1$, $H_n = G_n - \bar{G}_{n+1}$ for every $n (\geq 1)$. By Lemma 3, the family $\{\bar{H}_n\}_{n=0}^\infty$ is a locally finite, star-finite countable, regularly closed covering of S . Then, by the above assumption of the space S , the covering $\{\bar{H}_n\}_{n=0}^\infty$ is a finite covering. Hence, there is N such that for every $n \geq N$, $\bar{H}_n = \phi$ i.e. $H_n = \phi$ from which we have $G_n = \bar{G}_{n+1}$. In the proof of Lemma 3, $\bar{G}_n - G_n \subset \bar{H}_n$ for every $n (\geq 1)$, then we have $\bar{G}_n - G_n = \phi$ i.e. $\bar{G}_n = G_n$ for every $n \geq N$. Therefore, we have $\bar{G}_N = G_N = \bar{G}_{N+1} = G_{N+1} = \dots$. Then we have $\bigcap_{n=1}^\infty G_n = G_N$. From $\bigcap_{n=1}^\infty G_n = \phi$, we have $G_N = \phi$. Then for any family $\{G_n\}_{n=1}^\infty$ of nonvoid open sets such that $G_n \supset \bar{G}_{n+1}$ for every n , we must have $\bigcap_{n=1}^\infty G_n \neq \phi$, that is, the space S must be countably R -closed.

Lemma 4. Let $\{\bar{O}_\alpha\}_{\alpha \in A}$ be a locally finite, star-finite, regularly closed covering of any topological space S . We have the following properties.

- (i) The family $\{\bar{O}_\alpha\}_{\alpha \in A}$ is decomposed into components $\{K_\gamma\}_{\gamma \in \Gamma}$, where $K_\gamma = \{\bar{O}_\alpha\}_{\alpha \in A_\gamma}$ and $K_\gamma (\gamma \in \Gamma)$ are pairwise disjoint.
- (ii) The family $K_\gamma = \{\bar{O}_\alpha\}_{\alpha \in A_\gamma}$ is a countable family for every γ .
- (iii) The union $R_\gamma = \bigcup_{\alpha \in A_\gamma} \bar{O}_\alpha$ is open and closed for every γ .
- (iv) The family $\{R_\gamma\}_{\gamma \in \Gamma}$ of open and closed sets is a discrete covering of S .

Proof of (i). Let $\bar{O}_\alpha$ be an arbitrary member of $\{\bar{O}_\alpha\}_{\alpha \in A}$. Let us define $S^n(\bar{O}_\alpha)$ for $n=1, 2, \dots$ as follows:

$$\begin{aligned} S^1(\bar{O}_\alpha) &= \bigcup_\beta \{\bar{O}_\beta; \bar{O}_\beta \cap \bar{O}_\alpha \neq \phi, \beta \in A\}, \\ S^{n+1}(\bar{O}_\alpha) &= \bigcup_\beta \{\bar{O}_\beta; \bar{O}_\beta \cap S^n(\bar{O}_\alpha) \neq \phi, \beta \in A\} \quad \text{for every } n. \end{aligned}$$

For any $\bar{O}_\alpha$ and $\bar{O}_\beta$, if there is n such that $\bar{O}_\beta \subset S^n(\bar{O}_\alpha)$, we denote $\bar{O}_\beta \sim \bar{O}_\alpha$, which is an equivalence relation. Therefore the family $\{\bar{O}_\beta\}_{\beta \in A}$ is decomposed into components $\{K_\gamma\}_{\gamma \in \Gamma}$ where $K_\gamma = \{\bar{O}_\beta\}_{\beta \in A_\gamma}$ and $K_\gamma \cap K_{\gamma'} = \phi$ for any γ and $\gamma' (\gamma \neq \gamma')$.

Lemma 5. In any countably R -closed space S , if a subset E is open and closed, then the subspace E is countably R -closed. (See Z. Frolik [1], p. 217.)

Lemma 6. In a countably R -closed space S , any discrete open covering is a finite covering.

Proof that (3)→(4). Let S be a topological space having the property (3). By the implication (3)→(1), which has been shown, S is countably R -closed. Let $\{\bar{O}_\alpha\}_{\alpha \in A}$ be a locally finite, star-finite, regularly closed covering of S . By Lemma 4, the family $\{R_\gamma\}_{\gamma \in \Gamma}$ is a discrete covering of S composed of open and closed sets. By Lemma 6, the family $\{R_\gamma\}_{\gamma \in \Gamma}$ is a finite covering of S . Since the set R_γ is open and closed for every γ , by Lemma 5, the subspace R_γ is countably R -closed space. By Lemma 4, for every $\gamma \in \Gamma$, the family $K_\gamma = \{\bar{O}_\alpha\}_{\alpha \in A_\gamma}$ is a locally finite, star-finite, countable, regularly closed covering of the countably R -closed space R_γ . By the implications (1)→(2)→(3), which have been shown, the family $K_\gamma = \{\bar{O}_\alpha\}_{\alpha \in A_\gamma}$ is a finite covering of R_γ . Therefore the covering $\{\bar{O}_\alpha\}_{\alpha \in A}$ is a finite covering of S .

Proof that (4)→(3). It is obvious.

Proof that (4)→(5). From the proposition and the implications (1)⇔(3)⇔(4), the implications (4)⇔(5) are obvious. We shall prove directly that (4)→(5). Let S be a topological space which has the property (4) and let $\{O_\alpha\}_{\alpha \in A}$ be a star-finite open covering of S . It is obvious that the family $\{\bar{O}_\alpha\}_{\alpha \in A}$ is a locally finite, star-finite, regularly closed covering of S , from which, by the property (4), the family $\{\bar{O}_\alpha\}_{\alpha \in A}$ is finite i.e. the family $\{O_\alpha\}_{\alpha \in A}$ is finite, since $\bar{O}_\alpha \subset \bigcup_\beta \{O_\beta; O_\beta \cap O_\alpha \neq \emptyset, \beta \in A\}$ for every $\alpha \in A$.

Lemma 7. *Let $\{\bar{O}_\alpha\}_{\alpha \in A}$ be a locally finite, star-finite, regularly closed covering of any topological space S and let $S(\bar{O}_\alpha) = \bigcup_\beta \{\bar{O}_\beta; \bar{O}_\beta \cap \bar{O}_\alpha \neq \emptyset, \beta \in A\}$ for every $\bar{O}_\alpha$, then we have $\bar{O}_\alpha \subset \text{Int } S(\bar{O}_\alpha)$.*

Proof that (5)→(4). We shall prove directly that (5)→(4). Let S be a topological space which satisfies the condition (5) and let $\{\bar{O}_\alpha\}_{\alpha \in A}$ be a locally finite, star-finite, regularly closed covering of S . Put $G_\alpha = \text{Int } S(\bar{O}_\alpha)$ for every $\alpha \in A$. It is easily proved that the family $\{S(\bar{O}_\alpha)\}_{\alpha \in A}$ is a locally finite, star-finite, regularly closed covering of the space S . It is also easily proved that the family $\{G_\alpha\}_{\alpha \in A}$ is star-finite. By Lemma 7, $G_\alpha \supset \bar{O}_\alpha$ for every α , then $\{G_\alpha\}_{\alpha \in A}$ is a star-finite open covering of S . By the condition (5), the covering $\{G_\alpha\}_{\alpha \in A}$ is finite. Since $S(\bar{O}_\alpha) \supset G_\alpha$, $\{S(\bar{O}_\alpha)\}_{\alpha \in A}$ is a finite covering. Being star-finite, $\{\bar{O}_\alpha\}_{\alpha \in A}$ is a finite covering.

References

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- [2] K. Iséki: Generalisation of the notion of compactness. *Rev. Math. Pures Appl.*, **6**, 31–63 (1961).