185. δ_p and Countably Paracompact Spaces

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1971)

In [3], Mack defines the term δ -normal, and proves that if I is the closed unit interval, then a space X is countably paracompact if and only if $X \times I$ is δ -normal. In this paper we define the term δ_p which is stronger than δ -normal, but is strictly weaker than countable paracompactness, and is strictly weaker than normality; and we prove the following:

Theorem 1. *The following are equivalent for a space X*

- (i) *The space X is countably paracompact.*
- (ii) *The space X is δ_p and countably metacompact.*
- (iii) *The space X is δ_p and every countable open cover of X has a countable semi-refinement of closed sets.*
- (iv) *If C is a countable open cover of X , then there exists a countable collection $L = \{L_i | i = 1, 2, \dots\}$ of open refinements of C such that for each $x \in X$ there is some L_i that is locally finite with respect to x .*
- (v) *If I is the closed unit interval, then $X \times I$ is δ_p .*

We observe that (ii) of the above theorem is a slight generalization of a condition proven by Dowker [2]; further; we point out that Theorem 1 in [3] is used in proving (v) of the above theorem.

Definition. If X is a space and C is an open cover of X , then L is a semi-refinement of C if each member of L is contained in the union of a finite subset of C .

Definition. If X is a space and L is a collection of subsets of X , then L is locally finite with respect to a subset A of X , if for each $x \in A$, there exists an open set V , $x \in V$, such that V intersects only finitely many members of L .

Definition. Let X be a space and let N be a cardinal number. Then X is called an N_p space, if for each open cover C , cardinality of C less than or equal N , there exists for each closed set F contained in any member of C , an open refinement of C that is locally finite with respect to F . In the special case when $N = \aleph_0$, we will denote N_p by δ_p .

For an infinite cardinal N , a topological space is N -normal if each pair of disjoint closed sets, one of which is a regular G_N -set, have disjoint neighborhoods [3]. A set B is called a regular G_N -set if it is the intersection of at most N closed sets whose interiors contain B [3].

For $N = \aleph_0$, N -normal is denoted by δ -normal and a regular G_N set by regular G_δ set.

A space X is called countably metacompact if every countable open cover of X has a point finite open refinement. The terminology of [4] is followed except that we shall use $V(x)$, $N(x)$, etc. (resp., $V(A)$, $N(A)$, etc.) to denote open sets containing the point x (resp., the subset A).

At the end of the paper, we give an example of a δ -normal countably metacompact space that is not countably paracompact, thus, by Theorem 1, a δ -normal space that is not δ_p .

Theorem 2. *If N is a cardinal number, then we have the following:*

- (i) *Every paracompact space is N_p .*
- (ii) *Each normal space is N_p .*
- (iii) *Each N_p space is N -normal.*

Proof of Theorem 2. The proof of (i) is clear by the definition. To show (ii), let $C = \{G_\alpha \mid \alpha \in A\}$ be an open cover of a normal space X , and suppose a closed set $F_\lambda \subset G_\lambda$ for some $\lambda \in A$. Then there exists $V(F_\lambda)$ such that $\overline{V(F_\lambda)} \subset G_\lambda$. If we let $C_1 = \{G_\lambda\} \cup \{(X - \overline{V(F_\lambda)}) \cap G_\alpha \mid \alpha \in A\}$, then C_1 is an open refinement of C that is locally finite with respect to F_λ . Thus, X is an N_p space for all cardinal N .

Suppose N is a cardinal number, and X is an N_p space. Assume F and K are disjoint closed sets with F a regular G_N -set. By definition, $F = \bigcap \{\overline{W_\alpha(F)} \mid \alpha \in A, \text{ card. } A \leq N\}$. Since $F \subset X - K$ and since $C = \{X - K\} \cup \{X - \overline{W_\alpha(F)} \mid \alpha \in A\}$ is an open cover of X , there exists an open refinement $\{V_\lambda \mid \lambda \in A\}$ of C that is locally finite with respect to F . For each $x \in F$, there exists $N(x)$ such that $N(x) \cap V_\lambda \neq \emptyset$ for only finitely many $\lambda \in A$. By letting $L_x = \{V_\lambda \mid V_\lambda \cap N(x) \neq \emptyset \text{ and } V_\lambda \subset X - \overline{W_\alpha(F)} \text{ for some } \alpha \in A\}$, there exists $H(x)$ such that $H(x) \cap (\bigcup L_x) = \emptyset$. Put $M(x) = N(x) \cap H(x)$; then $M(x) \cap (\bigcup \{V_\lambda \mid \lambda \in A \text{ and } V_\lambda \subset X - \overline{W_\alpha(F)} \text{ for some } \alpha \in A\}) = \emptyset$. If we let $O(F) = \bigcup \{M(x) \mid x \in F\}$, then $O(F)$ is open and contains F . Further, if $V_\lambda \subset X - \overline{W_\alpha(F)}$, $\lambda \in A$ and $\alpha \in A$, then $O(F) \cap V_\lambda = \emptyset$. Hence $\overline{O(F)} \cap (\bigcup \{V_\lambda \mid \lambda \in A \text{ and } V_\lambda \subset X - \overline{W_\alpha(F)}\}) = \emptyset$; and therefore, $\overline{O(F)} \subset X - K$. It follows that X is N -normal.

Definition. For each $i = 1, \dots, n$, let L_i be a collection of sets. Then $\bigcap_{i=1}^n L_i = \{\bigcap_{i=1}^n (V_{(i,\alpha)}) \mid V_{(i,\alpha)} \in L_i\}$.

Lemma 3. *Let X be a space, and let C be a countable open cover of X . If there exists a countable collection $L = \{L_i \mid i = 1, 2, \dots\}$ of open refinements of C such that for each $x \in X$, there exists some L_i that is locally finite with respect to x , then there exists an open locally finite refinement of C .*

Proof. Let $C = \{G_i \mid i = 1, 2, \dots\}$ be a countable open cover of X , and let $L = \{L_i \mid i = 1, 2, \dots\}$ be a countable collection of open refinements

of C such that for each $x \in X$ there exists some L_i that is locally finite with respect to x . For each positive integer n , let $L^n = \{V \mid V \in \bigcap_{i=1}^n L_i, V \not\subset \bigcup_{i=1}^n (G_i)\}$; and let $A_n = \{x \mid x \in X \text{ and there exists } M(x), M(x) \cap (\bigcup L^n) = \emptyset\}$. To show that $\bigcup_{i=1}^\infty (A_i) = X$, let $x \in X$. Then there exists $N(x)$ and there exists a positive integer k such that $N(x)$ intersects only finitely many members of L_k ; hence, there exists a positive integer K such that $V \in L_k, N(x) \cap V \neq \emptyset$ implies $V \subset \bigcup_{i=1}^K (G_i)$. Thus, $N(x) \cap (\bigcup L^K) = \emptyset$, and therefore, $x \in A_K$. If n is a positive integer, there exists $W(A_n)$ such that $\overline{W(A_n)} \subset \bigcup_{i=1}^n (G_i)$. Now $H = \{G_1\} \cup \{G_n - \bigcup_{i=1}^{n-1} \overline{W(A_i)} \mid n = 1, 2, \dots\}$ is an open locally finite refinement of C .

Proof of Theorem 1. The proof that (i) implies (ii) is clear, (ii) implies (iii) is not very difficult; and (iv) implies (i) follows from Lemma 3.

To see that (iii) implies (iv), let $C = \{G_i \mid i = 1, 2, \dots\}$ be a countable open cover of X , and let $\{F_j \mid j = 1, 2, \dots\}$ be a countable closed semi-refinement of C . Then for each positive integer j , there exists a finite union $M_j = \bigcup_{k=1}^n (G_{(k,j)})$ of subsets of C such that $F_j \subset M_j$. If we let $C_j = \{M_j\} \cup C$, then there exists an open refinement H of C_j that is locally finite with respect to F_j . By letting $L_j = \{V \mid V \in H \text{ and } V \subset G_i \text{ for some } G_i \in C\} \cup \{V \cap G_{(k,j)} \mid k = 1, \dots, n, M_j = \bigcup_{k=1}^n G_{(k,j)}, V \subset M_j \text{ and } V \in H\}$, L_j is an open refinement of C that is locally finite with respect to F_j . Therefore, a desired collection is $L = \{L_j \mid j = 1, 2, \dots\}$.

That (i) implies (v) follows from Theorem 1 [2], and (v) implies (i) follows from Theorem 2 and Theorem 1 [3].

Comment. That every regular Lindelof space is paracompact follows from (iv) of Theorem 1.

A space X is defined to be point (countably) paracompact if for each (countable) open cover C of X , there exists for each $x \in X$ an open refinement of C that is locally finite with respect to x .

Remark. It is clear that each $T_1 \delta_p$ space is point countably paracompact, and that if X is a T_1 space that is an N_p space for each cardinal N , then X is point paracompact. By Theorem 2 [1], each T_2 point paracompact space is regular. Thus, if X is a T_2 space that is an N_p space for each infinite cardinal N , then X is regular, and the following proposition is obtained.

Proposition 4. *A T_2 space is normal if and only if it is an N_p space for every infinite cardinal N .*

Corollary 5. *Each T_2 paracompact space is normal.*

Example. There exists a δ -normal countably metacompact space that is not countably paracompact.

Let $X = \{(1/n, 0) \mid n = 1, 2, \dots\} \cup \{(1, 0)\} \cup \{(1/n, 1) \mid n = 1, 2, \dots\}$. The basic open sets are $\{(1/n, 1), (1/n, 0)\}$, $\{(1/n, 0)\}$, and $\{(1/n, 0) \mid n = 1, 2,$

$\dots\} \cup \{(1, 0)\}$. It is easy to see that the space is countably metacompact but not countably paracompact. The only regular G_δ subsets of X are X and the empty set. Therefore x is δ -normal.

References

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