205. On a Non-linear Volterra Integral Equation with Singular Kernel

By Takashi KANAZAWA

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 13, 1971)

In the present paper we consider the solution $y(x)$ of the non-linear Volterra integral equation

$$(1) \quad y(x) = f(x) + \int_0^x p(x, t) k(x, t, y(t)) dt$$

where $p(x, t)$ is supposed to be unbounded in the region of integration.

Examples. $p(x, t) = (x - t)^{-1/2}$, or $p(x, t) = t(x^2 - t^2)^{-1/2}$.

Evans [1] studied a similar problem using the convolution. Our treatment below is more elementary than his. We also consider the continuity and differentiability with respect to a parameter of solutions of (1) when it contains a parameter.

1. Existence theorem. In equation (1) we shall assume the four conditions:

(a) $f(x)$ is continuous in the interval I_a ,

$$I_a = \{x \mid 0 \leq x \leq a\};$$

(b) $k(x, t, y)$ is continuous in the region Δ ,

where $\Delta = \{(x, t, y) \mid 0 \leq t \leq x \leq a, |y - f(x)| \leq b\}$,

$$\sup_{0 \leq t \leq x \leq a} k(x, t, f(x)) = K,$$

$k(x, t, y)$ satisfies a Lipschitz condition:

$$|k(x, t, y_1) - k(x, t, y_2)| \leq L|y_1 - y_2|;$$

(c) $\int_0^x |p(x, t)| dt \leq M < \infty \quad (0 \leq x \leq a);$

(d) for any $\varepsilon > 0$, there exists $\delta > 0$, independent of x and α , such that

$$\int_\alpha^{\alpha+\delta} |p(x, t)| dt < \varepsilon \quad \text{for all } 0 \leq \alpha \leq x - \delta.$$

Theorem 1. Under the conditions (a), (b), (c), (d), equation (1) has a unique continuous solution on the interval $0 \leq x \leq h$, where h is determined as follows:

for any ρ , $0 < \rho < 1$, let $P = \min \left(\frac{\rho}{L}, \frac{b}{K} \right)$ and then let $h = \min(r, a)$,

where r is determined by

$$\int_0^x |p(x, t)| dt \leq P \quad (0 \leq x \leq r).$$

Proof. For $n = 1, 2, \dots$, let us put

$$y_n(x) = f(x) + \int_0^x p(x, t)k(x, t, y_{n-1}(t))dt,$$

where $y_0(x) = f(x)$.

Then, by our determination of P and h , $y_n(x)$ is defined in I_h and satisfies the inequality

$$|y_n(x) - f(x)| \leq b \quad (n=1, 2, \dots).$$

By Lipschitz condition, for $x \in I_h$,

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leq L \int_0^x |p(x, t)| |y_n(t) - y_{n-1}(t)| dt \\ &\leq L \sup_{x \in I_h} |y_n(x) - y_{n-1}(x)| \int_0^x |p(x, t)| dt \\ &\leq LP \sup_{x \in I_h} |y_n(x) - y_{n-1}(x)|. \end{aligned}$$

Hence we have

$$|y_{n+1}(x) - y_n(x)| \leq b(LP)^n \leq b\rho^n \quad (n=0, 1, 2, \dots).$$

Since $0 < \rho < 1$, the sequence $\{y_n(x)\}$ is uniformly convergent in I_h and $y(x) = \lim_{n \rightarrow \infty} y_n(x)$ is a continuous solution of equation (1).

The uniqueness follows from the Lipschitz condition on $k(x, t, y)$.

2. Prolongation of solution.

Theorem 2. *The solution curve of equation (1) in Theorem 1 can be prolonged to the terminal point $x=a$ of the interval I_a , when $k(x, t, y)$ is defined and continuous in $0 \leq t \leq x \leq a$, $|y| < \infty$.*

Proof. Let $y = \bar{y}(x)$ be the solution of (1) in the interval I_h , $0 \leq x \leq h (< a)$, where $h = \min(r, a)$ and r is determined by

$$\int_0^x |p(x, t)| dt \leq P = \frac{\rho}{L} \quad (0 \leq x \leq r).$$

Then, from Theorem 1 we know that the equation

$$y(x) = f(x) + \int_0^h p(x, t)k(x, t, \bar{y}(t))dt + \int_h^x p(x, t)k(x, t, y(t))dt$$

has a continuous solution $y = \bar{\bar{y}}(x)$ in the interval $h \leq x \leq 2h (\leq a)$, because $\int_0^h p(x, t)k(x, t, \bar{y}(t))dt$ is bounded and continuous.

The function

$$y(x) = \begin{cases} \bar{y}(x) & 0 \leq x \leq h \\ \bar{\bar{y}}(x) & h \leq x \leq 2h \end{cases}$$

is the solution of (1) in the interval I_{2h} .

Repeating the same procedure finite times, we can reach to $x=a$.

3. Equation containing a parameter. Let us consider the integral equation

$$(2) \quad y(x) = f(x, \lambda) + \int_0^x p(x, t)k(x, t, y(t), \lambda)dt$$

containing a parameter λ .

Theorem 3. *In equation (2), we shall assume the following*

conditions:

(a') $f(x, \lambda)$ is continuous in the region

$$\{(x, \lambda) | 0 \leq x \leq a, |\lambda| \leq l\};$$

(b') $k(x, t, y, \lambda)$ is bounded continuous in the region

$$\{(x, t, y, \lambda) | 0 \leq t \leq x \leq a, |y - f(x, \lambda)| \leq b, |\lambda| \leq l\}$$

and satisfies a Lipschitz condition

$$|k(x, t, y_1, \lambda) - k(x, t, y_2, \lambda)| \leq L |y_1 - y_2|;$$

(c) $\int_0^x |p(x, t)| dt \leq M < \infty \quad (0 \leq x \leq a);$

(d') for any $\varepsilon > 0$, there exists a $\delta > 0$, independent of α and x , such that

$$\int_\alpha^{\alpha+\delta} |p(x, t)| dt < \varepsilon \quad \text{for all } 0 \leq \alpha \leq x - \delta.$$

Further suppose that for any λ ($|\lambda| \leq l$) there exists a (unique) solution of (2) in the interval $I_a = \{x | 0 \leq x \leq a\}$.

Then the solution $y = y(x, \lambda)$ is continuous with respect to the parameter λ in the region $\{(x, \lambda) | 0 \leq x \leq a, |\lambda| \leq l\}$.

Proof. For any ρ , $0 < \rho < 1$, let $h = \min(r, a)$, where r is determined by $\int_0^x |p(x, t)| dt \leq \frac{\rho}{L}$ ($0 \leq x \leq r$).

First we consider the solution $y(x, \lambda)$ in the region $\{(x, \lambda) | 0 \leq x \leq h, |\lambda| \leq l\}$.

For any $\lambda + \Delta\lambda$, $|\lambda + \Delta\lambda| < 1$, we have

$$\begin{aligned} & y(x, \lambda + \Delta\lambda) - y(x, \lambda) \\ &= f(x, \lambda + \Delta\lambda) - f(x, \lambda) \\ &+ \int_0^x p(x, t) \{k(x, t, y(t, \lambda + \Delta\lambda), \lambda + \Delta\lambda) - k(x, t, y(t, \lambda), \lambda + \Delta\lambda)\} dt \\ &+ \int_0^x p(x, t) \{k(x, t, y(t, \lambda), \lambda + \Delta\lambda) - k(x, t, y(t, \lambda), \lambda)\} dt, \end{aligned}$$

and therefore

$$(3) \quad \begin{aligned} & |y(x, \lambda + \Delta\lambda) - y(x, \lambda)| \\ & \leq \delta_1(\Delta\lambda) + L \int_0^x |p(x, t)| |y(t, \lambda + \Delta\lambda) - y(t, \lambda)| dt + M \delta_2(\Delta\lambda), \end{aligned}$$

where $|f(x, \lambda + \Delta\lambda) - f(x, \lambda)| \leq \delta_1(\Delta\lambda)$,

$$|k(x, t, y, \lambda + \Delta\lambda) - k(x, t, y, \lambda)| \leq \delta_2(\Delta\lambda).$$

Let

$$\delta_1(\Delta\lambda) + M \delta_2(\Delta\lambda) = \delta(\Delta\lambda),$$

then, from assumptions (a') and (b')

$$\delta(\Delta\lambda) \rightarrow 0 \quad \text{as } \Delta\lambda \rightarrow 0.$$

Let

$$\sup_{(x, \lambda) \in I_h \times A} |y(x, \lambda + \Delta\lambda) - y(x, \lambda)| = \delta(y),$$

then from (3)

$$\delta(y) \leq \delta(\Delta\lambda) + \rho \delta(y)$$

or

$$\delta(y) \leq \delta(\Delta\lambda)/(1-\rho).$$

Therefore, when $\Delta\lambda \rightarrow 0$, $\delta(\Delta\lambda) \rightarrow 0$ and $\delta(y) \rightarrow 0$, that is, $y(x, \lambda)$ is continuous with respect to λ .

Applying the same argument successively, we can prove the continuity of $y(x, \lambda)$ with respect to λ on the whole interval I_a . Q.E.D.

From Theorem 3, we have evidently

Theorem 4. *Suppose the equation (2) satisfies the following conditions:*

(a'') $f(x, \lambda)$, $\frac{\partial}{\partial \lambda} f(x, \lambda)$ are continuous in the region

$$\{(x, \lambda) | 0 \leq x \leq a, |\lambda - \lambda_0| \leq l\}.$$

(b'') $k(x, t, y, \lambda)$, $\frac{\partial}{\partial y} k(x, t, y, \lambda)$, $\frac{\partial}{\partial \lambda} k(x, t, y, \lambda)$

are continuous in the region

$$\{(x, t, y, \lambda) | 0 \leq x \leq t \leq a, |y - f(x, \lambda)| \leq b, |\lambda - \lambda_0| \leq l\};$$

(c) $\int_0^x |p(x, t)| dt \leq M < \infty \quad (0 \leq x \leq a)$

(d) for any $\varepsilon > 0$, there exists a $\delta > 0$, independent of α, x, λ , such that

$$\int_{\alpha}^{\alpha+\delta} |p(x, t)| dt < \varepsilon \quad \text{for all } 0 \leq \alpha \leq x - \delta.$$

Further suppose that for any λ , $|\lambda - \lambda_0| \leq l$, there exists a (unique) solution $y(x, \lambda)$ of (2) in the interval I_a .

Then the solution $y(x, \lambda)$ is continuously differentiable with respect to λ at $\lambda = \lambda_0$.

Acknowledgement. The author wishes to thank Professor M. Urabe for his kind guidance.

Reference

- [1] G. C. Evans: Volterra's integral equation of the second kind with discontinuous kernel. Trans. Amer. Math. Soc., **11**, 393-413 (1910).