206. Remark on Fixed Point of k-regular Mappings

By Haruo Maki
Department of Mathematics, Wakayama University, Wakayama


The main purpose of this paper is to answer the question raised in [4]. The dilation $D_k$ of Euclidean $n$-space $\mathbb{R}^n$ defined by $x \mapsto kx$ for some $k \in (0, 1)$ can be extended uniquely to the $n$-sphere, $S^n = \mathbb{R}^n \cup \{\infty\}$. If $h$ is a homeomorphism of $S^n$ of the same topological type as $D_k$, then $h$ is regular except at two points. Kérekjarto [6], Homma and Kinoshita [2] showed the converse for $n=2$, $n=3$ respectively. Husch [3] extended Homma and Kinoshita’s result for $n \geq 6$. He [4] considered the topological characterization of the dilation in a separable infinite dimensional Fréchet space $E$ (i.e. in a separable infinite dimensional locally convex complete linear metric space).

In [4], Husch has the following theorems. Let $h$ be a homeomorphism of $E$ (with metric $d$) onto itself.

Theorem (Husch [4]). Suppose that $h$ is $k$-regular at each point of $E$, $0 < k < 1$ (i.e. for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(h^n(x), h^n(y)) < k^n \varepsilon$ for each integer $n$).

(1) ([4], Proposition 6, p. 4) $h$ has at most one fixed point.

(2) ([4], Theorem 1, p. 2) If the fixed point set of $h$, $\text{Fix}(h)$, is not empty, then $h$ has the topological type of a dilation $D_k$.

(3) ([4], Theorem 2, p. 2) If $\text{Fix}(h)$ is empty, then $h$ has the topological type of a translation.

In this paper we prove the following:

Theorem 1. If $h$ is $k$-regular at each point of $E$, $0 < k < 1$, then $h$ has a unique fixed point.

Hence we can eliminate the hypothesis that $\text{Fix}(h)$ be a non empty set in Husch’s result (2).

Every separable infinite dimensional Fréchet space $E$ is homeomorphic to the countable infinite product of lines [1]. Hence $E$ is connected metric space. Thus we only show the following:

Lemma 2. Let $h$ be a $k$-regular mapping, $(0 < k < 1)$, of a complete, connected metric space $X$ onto itself. Then $h$ has a unique fixed point.

Before starting the proof, we recall the following definitions and some properties [5]. Let $h$ be a continuous mapping in a metric space $X$. If for each $\varepsilon > 0$, there exists $n \in 1^*$ (positive integers) such that $d(h^n(x), h^n(y)) < \varepsilon$ for all $m \geq n$, 

\[ d(h^n(x), h^n(y)) < \varepsilon \quad \text{for all } m \geq n, \]
then $x$ and $y$ are said to be asymptotic under $h$. (Abbreviate $x \sim y$).
Then $\sim$ is an equivalence relation on $X$. Let $X_h$ be the set of all equivalence classes. $\hat{x}$ denotes the equivalence class of $x \in X$. The induced mapping $\hat{h} : X_h \to X_h$ is well defined as follows. For each $\hat{x} \in X_h$, $\hat{h}(\hat{x}) = \hat{h}(x)$. Then we have the following theorems.

**Theorem 3** (Kashiwagi and Maki [5], Theorem 12, p. 7). *Let $X$ be a complete metric space. Then the continuous mapping $h$ has a unique fixed point if and only if the induced mapping $\hat{h}$ has a unique fixed point.*

**Theorem 4** ([5], Theorem 13, p. 7). *Let all assumptions of Theorem 3 hold. If $X_h$ is a singleton, then $h$ has a unique fixed point.*

**Proof of Lemma 2.** Since $h$ is $k$-regular at each point $x$, there exists a $\delta(x)$-neighbourhood $B_x(\delta(x))$, with center $x$ and radius $\delta(x)$ such that

if $\forall y \in B_x(\delta(x))$, then $x \sim y$.

Let $x$ be any point of $X$. By the above discussion, $\hat{x}$ is open in $X$. And $\hat{x}$ is not empty. Note that $\hat{x}$ is a closed set in $X$. For suppose $\{x_n\}$ is a sequence in $\hat{x}$ such that

$$x_n \to a$$

as $n \to +\infty$.

Since $h$ is $k$-regular at $a$, then there exists an integer $N$ such that

$$x_n \sim a \quad \text{for all } n > N. \quad x, x_n \in \hat{x} \text{ implies } x_n \sim x \quad \text{for } n.$$

Hence we have $x \sim a$, $a \in \hat{x}$. This implies $\hat{x}$ is closed in $X$. Since $X$ is connected, $X = \hat{x}$. Hence $X_h$ is a singleton. With the use of Theorem 4, the proof is complete. Q.E.D.

**Remark 1.** Theorem 1 is the answer to the question raised in [4]. The hypothesis that $\text{Fix} (h)$ be empty can never be satisfied in Husch's result (3). Hence that lines should be deleted from the theorem (Theorem 2 [4]).

Now, suppose that there exists an everywhere dense subset $Y$ of $X$. We have the following:

**Theorem 5.** *Let $f$ be a $k$-regular mapping of a complete metric space $X$ onto itself. If $Y_f$ is a singleton, then $f$ has a unique fixed point.*

**Proof.** We show that $X_f$ is a singleton. Let $x$ be any point of $Y$, $y$ any point of $X - Y$. Then there exists a sequence $\{x_n\}$ of $Y$ such that

$$\lim_{n \to \infty} x_n = y.$$

Since $f$ is $k$-regular, $x_n \sim y$ for some integer $n$. Clearly $x_n \sim x$. Hence $x \sim y$ if $x \in Y$, $y \in X - Y$. Now let $x$, $y$ be any points of $X - Y$. Then there exist the sequences $\{x_n\}$, $\{y_n\}$ such that

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y.$$

Since $x_n$, $y_n \in Y$, $x \in X - Y$, then $x_n \sim x$ and $y_m \sim y$ for some integers $n$, $m$. Since $Y_f$ is a singleton, we have $x \sim y$. This implies $X_f$ is a
Remark 2. If one replaces the condition that $f$ is $k$-regular with the condition that $f$ is continuous, then the resulting proposition need not be true, as the following example shows. Define $f$ on the interval $X=[-\sqrt{2},+\infty)$ as follows:

$$f(x)=\begin{cases} 2x+\sqrt{2} & \text{if } -\sqrt{2} \leq x \leq \sqrt{2}, \\ x/2+5/\sqrt{2} & \text{if } x > \sqrt{2}. \end{cases}$$

$f$ is not $k$-regular, $0<k<1$, and $Y_f$ is a singleton where $Y=Q \cap X$. But $f$ has two fixed points.

In the end of this paper, we give another application of Theorem 4, which treat a subject of Kannan's fixed point theorem in metric space.

**Theorem 6.** Let $X$ be a complete metric space. Let $f$ be a continuous mapping of $X$ into itself such that

$$d(f^n(x), f^{n+1}(x)) \leq \alpha d(x, f^n(x)) + \beta d(y, f^n(y)) + \gamma d(x, y)$$

where $x, y \in X$ and $0<\alpha+\beta+\gamma<1$, $0\leq \alpha$, $0\leq \beta<1$, $0\leq \gamma<1$. Then $f$ has a unique fixed point.

**Proof.** Let $x$, $y$ be any point of $X$. In order to complete the proof, we see $x \sim y$. For all $n$ we have

$$d(f^n(x), f^{n+1}(x)) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^n d(x, f^n(x)).$$

Hence we have

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \alpha \left(\frac{\alpha+\gamma}{1-\beta}\right)^n d(x, f^n(x)) + \beta \left(\frac{\alpha+\gamma}{1-\beta}\right)^n d(y, f^n(y)) + \gamma d(f^n(x), f^{n}(y)).$$

By the induction,

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \left(\sum_{i=0}^{n} \gamma^i \left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-i}\right) \{\alpha d(x, f^n(x)) + \beta d(y, f^n(y))\} + \gamma^{n+1}d(x, y).$$

Let $B_n = \sum_{i=0}^{n} \gamma^i \left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-i}$. We have $\lim_{n \to \infty} B_n = 0$. Therefore

$$d(f^n(x), f^n(y)) \to 0 \quad \text{as } n \to +\infty.$$

This implies that $X_f$ is a singleton. By Theorem 4 the proof is complete.

**Q.E.D.**

Remark 3. By Theorem 6, we have the Banach's fixed point Theorem and Kannan's result [7].

Added in proof. Some changes need in Theorem 3 and Theorem 4. $h$ has a unique fixed point $\bar{x}$ with a Cauchy sequence $\{h^n(x)\}$, if and only if, $h$ has a unique fixed point. However, we assume that there exists a Cauchy sequence $\{h^n(x)\}$ for some $x$. In this case, if $X_h$ is a singleton, then $h$ has a unique fixed point. Therefore, in this case
Theorem 1, Lemma 2 and Theorem 5 are valid. Thus Remark 1 and two lines (p. 925, lines 24, 25) in this paper, should be deleted.

References