212. Some Nonlinear Evolution Equations of Second Order

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1. Introduction. Let $H$ and $W$ be two real separable Hilbert spaces and $V$ be a real separable reflexive Banach space with $V \subset W \subset H$. Let $V$ be dense in $W$ and in $H$ and the natural injections of $V$ into $W$ and of $W$ into $H$ be respectively continuous and compact. We identify $H$ with its dual:

$$V \subset W \subset H \subset W^* \subset V^*$$

where $W^*$ and $V^*$ are the duals of $W$ and $V$, respectively. The pairing between $V$ and $V^*$ is denoted by $(\cdot, \cdot)$ and that of $W$ and $W^*$ by $\langle \cdot, \cdot \rangle$.

We consider the following second order differential equation

$$u'' + A(u) + Bu' = f$$

with initial conditions

$$u(0) = u_0, \quad u'(0) = u_1,$$

where $u = u(t)$, $u' = du/dt$, $u'' = d^2u/dt^2$ and data $u_0$, $u_1$, $f$ are given.

Assume that the nonlinear operator $A: V \to V^*$ has the following properties:

1) $A$ is hemicontinuous and $\|A(u)\|_{V^*} \leq c \|u\|_V^{1/p}$, $p > 1$, $c > 0$.
2) $A$ is monotone, i.e., $(A(u) - A(v), u - v) \geq 0$, $u, v \in V$.
3) $(A(u), u) = \|u\|_V^2$.
4) $A(u)$ is Fréchet differentiable at every $u \in V$.
5) $A(u)$ is strongly homogeneous of degree $p - 1$ in the sense of Dubinskii [1], i.e., for every $u, \eta \in V$

$$(A'(u)\eta, u) = (A'(u)u, \eta) = (p - 1)(A(u), \eta)$$

where $A'(u)$ is a Fréchet derivative.

Let $B: W \to W^*$ be a bounded linear operator associated with a bounded symmetric bilinear form $b(\cdot, \cdot)$ on $W$, i.e.,

$$|b(u, v)| \leq \|u\|_W \|v\|_W, \quad b(u, v) = b(v, u),$$

$$b(u, v) = \langle Bu, v \rangle, \quad \forall u, v \in W,$$

such that

$$(1.4) \quad b(u, u) \geq \alpha \|u\|^p_W - \beta \|u\|^p_H, \quad \alpha, \beta > 0,$$

and that if $u_n \rightharpoonup u$ weakly in $W$ as $n \to \infty$,

$$(1.5) \quad \lim inf_n b(u_n, u_n) \geq b(u, u).$$

The main result of this note is the following theorem.

**Theorem 1.** Suppose that $u_0 \in V$, $u_1 \in H$ and $f \in L^2(0, T; H)$. Then there exists at least one function $u$ such that
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(1.6) \( u(t) \in L^\infty(0, T; V) \),
(1.7) \( u'(t) \in L^\infty(0, T; H) \cap L^1(0, T; W) \)
(1.8) \( u''(t) \in L^1(0, T; V^*) \)
and satisfies (1.1) and (1.2).

The proof of Theorem 1 is stated in Section 2. In Section 3, as applications, the existence of the weak solutions of the initial-Dirichlet boundary value problem for the equation of the form

\[
\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right)^{2p-1} - \Delta \frac{\partial u}{\partial t} = f,
\]

\( \Delta v = \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}, \quad p > 1, \)
will be established. When \( n = 1 \), the equation (1.9) was studied by Greenberg, MacCamy and Mizel [2] and Greenberg [3].

2. Proof of Theorem 1.

Lemma 1. For \( u(t) \in C([0, T]; V) \), we have

\[
\int_0^t (A(u(s)), u'(s)) ds = \frac{1}{p} (A(u(t)), u(t)) - \frac{1}{p} (A(u(0)), u(0))
\]

\( (\cdot , \cdot)_V = \frac{1}{p} \| u(t) \|^p - \frac{1}{p} \| u(0) \|^p. \)

Proof. By the chain rule, we have

\[
\frac{d}{dt} (A(u(t)), u(t)) = (A'(u(t))u'(t), u(t)) = (p-1)(A(u(t)), u'(t))
\]
since \( A(u) \) is strongly homogeneous of degree \( p - 1 \). Then we get

\[
\frac{d}{dt} (A(u(t)), u(t)) = p(A(u(t)), u'(t))
\]
which implies (2.1). q.e.d.

The following lemma can be found in [4].

Lemma 2. Let \( X \) be a reflexive separable Banach space. Then there exists a separable Hilbert space \( Y \), being dense in \( X \), such that the injection of \( Y \) into \( X \) is continuous.

Hence, we can construct a separable Hilbert space \( \tilde{H} \subset W \), being dense in \( V \), such that the injection of \( \tilde{H} \) into \( V \) is continuous. Then the injection of \( \tilde{H} \) into \( H \) is compact. Therefore we have

Lemma 3. The spectral problem:

\[
(w, v)_\tilde{H} = \lambda (w, v)_H, \quad \forall v \in \tilde{H},
\]
has the sequence of non zero solutions \( w_j \) corresponding to the sequence of eigenvalues \( \lambda_j \):

\[
(w, v)_\tilde{H} = \lambda_j (w, v)_H, \quad \forall v \in \tilde{H}, \quad \lambda_j > 0,
\]
where \( (\cdot , \cdot)_H \) and \( (\cdot , \cdot)_{\tilde{H}} \) are the scalar products in \( H \) and \( \tilde{H} \), respectively.

In order to prove Theorem 1, we shall employ the Galerkin’s method. We use the sequence of the functions \( w_j \) as the basis of \( \tilde{H} \).
We look for an approximate solution \( u_m(t) \) in the form:

\[
u_m(t) = \sum_{i=1}^{m} g_{i m}(t) w_i, \quad g_{i m}(t) \in C^0[0, T],
\]

where the unknown functions \( g_{i m} \) are determined by the following system of ordinary differential equations:

\[
(2.4) \quad (u_m''(t), w_j) + (A(u_m(t)), w_j) + b(u_m'(t), w_j) = (f(t), w_j) \quad 1 \leq j \leq m,
\]

with initial conditions:

\[
(2.5) \quad u_m(0) = u_{0m}, \quad u_{0m} = \sum_{i=1}^{m} \alpha_{i m} w_i \rightarrow u_0 \text{ in } V \text{ strongly as } m \rightarrow \infty,
\]

\[
(2.6) \quad u_m'(0) = u_{1m}, \quad u_{1m} = \sum_{i=1}^{m} \beta_{i m} w_i \rightarrow u_1 \text{ in } H \text{ strongly as } m \rightarrow \infty.
\]

Then we have

**Lemma 4.** There exists a constant \( c \) independent of \( m \), such that

\[
(2.7) \quad \|u_m\|_{L^m(0,T;V)} \leq c,
\]

\[
(2.8) \quad \|u'_m\|_{L^m(0,T;H) \cap L^2(0,T;W)} \leq c,
\]

\[
(2.9) \quad \|A(u_m)\|_{L^m(0,T;V^*)} \leq c,
\]

\[
(2.10) \quad \|B u'_m\|_{L^m(0,T;W^*)} \leq c,
\]

and

\[
(2.11) \quad \|u_m''\|_{L^m(0,T;\tilde{H}^*)} \leq c.
\]

**Proof.** Multiplication of the \( i \)-th equation in (2.4) by \( g_{i m} \), summation over \( i \) from 1 to \( m \), integration with respect to \( t \) and Lemma 1 give

\[
(2.12) \quad \frac{1}{2} \|u_m'(t)\|_{H}^2 + \frac{1}{p} \|u_m(t)\|_{W}^p + \int_0^t b(u_m'(s), u_m'(s))ds = \frac{1}{2} \|u_m(0)\|_{H}^2 + \frac{1}{p} \|u_m(0)\|_{W}^p + \int_0^t (f(s), u_m'(s))ds
\]

from which it follows that

\[
(2.13) \quad \frac{1}{2} \|u_m'(t)\|_{H}^2 + \frac{1}{p} \|u_m(t)\|_{W}^p + \alpha \int_0^t \|u_m'(s)\|_{W}^p ds \leq c \left( 1 + \int_0^t \|u_m'(s)\|_{H}^2 ds \right).
\]

The inequality (2.13) and our hypotheses on \( A \) and \( B \) yield (2.7)–(2.10).

Let \( P_m \) be the projection of \( H \rightarrow \{w_1, \ldots, w_m\} (= \text{the space spanned by } w_1, \ldots, w_m) \). Then we have \( P_m \in \mathcal{L}(\tilde{H}, \tilde{H}); \quad \|P_m\|_{\mathcal{L}(\tilde{H}, \tilde{H})} \leq c. \)

Since \( \tilde{H} \subset V \subset W \), we get

\[
\|P_m\|_{\mathcal{L}(\tilde{H}, V)} \leq c \quad \text{and} \quad \|P_m\|_{\mathcal{L}(\tilde{H}, W)} \leq c
\]

which imply

\[
\|P_m^*\|_{\mathcal{L}(V^*, \tilde{H}^*)} \leq c \quad \text{and} \quad \|P_m^*\|_{\mathcal{L}(W^*, \tilde{H}^*)} \leq c.
\]

The equation (2.4) may be written as

\[
u_m'' = -P_m^* A(u_m) - P_m^* B u'_m + P_m^* f,
\]

which assures (2.11).

From Lemma 4 we see that there exist a function \( u \) and a sub-
sequence \( u_m \) of \( u_m \) such that
\[
\begin{align*}
(2.14) \quad u_m & \rightharpoonup u & \text{in } L^\infty(0,T;V) \text{ weakly star}, \\
(2.15) \quad u_m' & \rightharpoonup u' & \text{in } L^\infty(0,T;H) \text{ weakly star and in } L^2(0,T;W) \text{ weakly}, \\
(2.16) \quad u_m'' & \rightharpoonup u'' & \text{in } L^2(0,T;\tilde{H}^*) \text{ weakly}, \\
(2.17) \quad u_m(T) & \rightharpoonup u(T) & \text{in } W \text{ weakly}, \\
(2.18) \quad u_m'(T) & \rightharpoonup u'(T) & \text{in } H \text{ weakly}, \\
(2.19) \quad A(u_m) & \rightharpoonup \chi & \text{in } L^\infty(0,T;V^*) \text{ weakly star},
\end{align*}
\]
and
\[
(2.20) \quad Bu_m' & \rightharpoonup Bu' & \text{in } L^2(0,T;W^*) \text{ weakly}.
\]

Since the injections of \( V \) into \( H \) and of \( W \) into \( H \) are compact, we can furthermore assume that
\[
\begin{align*}
(2.21) \quad u_m & \rightarrow u & \text{in } L^2(0,T;H) \text{ strongly}, \\
(2.22) \quad u_m(T) & \rightarrow u(T) & \text{in } H \text{ strongly} \\
\text{and} \\
(2.23) \quad u_m' & \rightarrow u' & \text{in } L^2(0,T;H) \text{ strongly}.
\end{align*}
\]

To show that the function \( u(t) \) is a solution of (1.1), (1.2), it is sufficient to prove that
\[
\chi = A(u).
\]

Multiplying (2.4) by an arbitrary smooth function \( a(t) \), integrating over \([0,T]\) and integrating the first term by parts, we have
\[
\begin{align*}
-\int_0^T (u'_m(t), \alpha'(t)w_j) \, dt + \int_0^T (A(u_m(t)), \alpha(t)w_j) \, dt \\
+ \int_0^T b(u'_m(t), \alpha(t)w_j) \, dt
\end{align*}
\]
\[
= \int_0^T (f(t), \alpha(t)w_j) \, dt + (u'_m(0), \alpha(0)w_j) - (u'_m(T), \alpha(T)w_j).
\]
Taking the limit of both sides with \( m = \mu, \ j \) fixed, we get
\[
\begin{align*}
-\int_0^T (u'(t), \alpha'(t)w_j) \, dt + \int_0^T (\chi, \alpha(t)w_j) \, dt + \int_0^T b(u'(t), \alpha(t)w_j) \, dt \\
= \int_0^T (f(t), \alpha(t)w_j) \, dt + (u', \alpha(0)w_j) - (u'(T), \alpha(T)w_j), \ \forall j,
\end{align*}
\]
which implies
\[
\begin{align*}
-\int_0^T (u'(t), \psi'(t)) \, dt + \int_0^T (\chi, \psi(t)) \, dt + \int_0^T b(u'(t), \psi(t)) \, dt \\
= \int_0^T (f(t), \psi(t)) \, dt + (u', \psi(0)) - (u'(T), \psi(T))
\end{align*}
\]
for any \( \psi \in G \), where \( G \) denotes a family of functions defined by
\[
G = \{ \psi | \psi \in L^2(0,T;V), \psi' \in L^2(0,T;H) \}.
\]
In particular, setting \( \psi = u \), we have
\[
\begin{align*}
-\int_0^T \|u'\|^H_T \, dt + \int_0^T (\chi, u) \, dt + \frac{1}{2} b(u(T), u(T)) - \frac{1}{2} b(u_0, u_0) \\
= \int_0^T (f, u) \, dt + (u_t, u_0) - (u'(T), u(T)).
\end{align*}
\]
The monotonicity of $A$ gives

$$X = \int_0^T (A(v), u_\rho - v) dt \geq 0, \quad \forall v \in L^\infty(0, T; V).$$

From (2.4) we have

$$\int_0^T (A(u_\rho), u_\rho) dt = \int_0^T \| u'_\rho \|_H^2 dt - \frac{1}{2} b(u_\rho(T), u_\rho(T))
+ \frac{1}{2} b(u_\rho(0), u_\rho(0)) + \int_0^T (f, u_\rho) dt
+ (u_\rho'(0), u_\rho(0)) - (u_\rho'(T), u_\rho(T))$$

from which it follows that

$$X = \int_0^T \| u'_\rho \|_H^2 dt - \int_0^T \frac{1}{2} b(u(T), u(T)) + \frac{1}{2} b(u_\rho(0), u_\rho(0))
+ \int_0^T (f, u_\rho) dt + (u_\rho'(0), u_\rho(0)) - (u_\rho'(T), u_\rho(T))
- \int_0^T (A(v), u_\rho - v) dt - \int_0^T (A(v), v) dt.$$

Hence, in virtue of (1.5) and (2.17) we get

$$\lim \inf_{\rho} X \leq \int_0^T \| u'_\rho \|_H^2 dt - \frac{1}{2} b(u(T), u(T)) + \frac{1}{2} b(u_\rho, u_\rho)
+ \int_0^T (f, u_\rho) dt + (u_\rho'(0), u_\rho(0)) - (u_\rho'(T), u_\rho(T))
- \int_0^T (\chi, v) dt.$$

Combining (2.26) with (2.28), we have

$$\int_0^T (\chi - A(v), u_\rho - v) dt \geq 0.$$

Then, a well-known argument of the theory of monotone operators gives

$$\chi = A(u).$$

From (1.1), we have

$$u'' = -A(u) - Bu' + f \in L^1(0, T; V^*).$$

This completes the proof of Theorem 1.

3. Some Examples. Let $Q$ be a bounded domain in $\mathbb{R}^n$ with a sufficient smooth boundary $\partial Q$. Points in $Q$ are denoted by $x = (x_1, \ldots, x_n)$ and the time variable is denoted by $t$. We consider the following initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right)^{p-1} - u \frac{\partial u}{\partial t} = f,$$

(3.2) $u(x, 0) = u_0(x), \quad \frac{\partial u(x, 0)}{\partial t} = u_1(x),$ 

(3.3) $u(x, t) = 0$ on $\partial Q \times [0, T]$,

where $f(x, t)$, $u_0(x)$ and $u_1(x)$ are given functions and $T$ is an arbitrary positive number.

Put
(3.4) \[ A(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right)^{2p-1} \]

and

(3.5) \[ b(u, v) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx. \]

If we take \( H=L^r(\Omega) \), \( W=W^{1,p}_0(\Omega) \) and \( V=W^{1,2p}_0(\Omega) \), we easily see that our hypotheses on \( A \) and \( B \) are satisfied. Furthermore the well-known theorem of Sobolev tells us that if

\[ r > 1 + \frac{n}{2} \frac{n}{2p} \]

then

\[ \tilde{H} = W^{\alpha,p}_{-}(\Omega) \subset W^{1,2p}_0(\Omega) \]

and the injection of \( W^{\alpha,p}_0(\Omega) \) into \( W^{1,2p}_0(\Omega) \) is continuous. Hence, we have

**Theorem 2.** For each \( f \in L^r(0, T; L^r(\Omega)) \), \( u_0 \in W^{1,2p}_0(\Omega) \), \( u_1 \in L^r(\Omega) \), the initial boundary value problem (3.1)–(3.3) has a solution \( u(x, t) \in L^\infty(0, T; W^{1,2p}_0(\Omega)) \) with

\[ \frac{\partial u(x, t)}{\partial t} \in L^\infty(0, T; L^r(\Omega)) \cap L^2(0, T; W^{1,2p}_0(\Omega)) \]

and

\[ \frac{\partial^2 u(x, t)}{\partial t^2} \in L^r(0, T; W^{-1,2p/2p-1}(\Omega)). \]

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References