216. A Remark on Semi-groups of Local Lipschitzians in Banach Space

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1. Introduction. Let X be a Banach space with the norm denoted by $\|\cdot\|$ and let C be a subset of X . A one-parameter family $\{T_t; 0 \leq t < +\infty\}$ of (nonlinear) operators of C into itself is called a *semi-group* on C if it satisfies the following conditions:

(i) $T_0 = I|_C$ (the identity mapping restricted to C) and $T_{t+s} = T_t T_s$ for $t, s \geq 0$;

(ii) For each fixed $x \in C$, $T_t x$ is strongly continuous in $t \geq 0$.

A (possibly) multiple-valued¹⁾ operator A (with the domain $D(A)$ and the range $R(A)$) in X is said to be a *D-operator* (in the terminology of Chambers and Oharu [1]) if it satisfies the following condition:

(D) There exists a non-negative function $\omega = \omega(r)$ on $(0, +\infty)$ such that $A|_{B_r} - \omega(r)I$ is dissipative for each $r > 0$; where $B_r = \{x \in X; \|x\| \leq r\}$.

The purpose of this paper is to give a sufficient condition in order that a D-operator A in X generate a semi-group on $\overline{D(A)}$ and show some examples. The condition is a modified version of that in [1].

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2. Generation of semi-groups. Our theorem reads:

Theorem 1. We assume that A is a D-operator in X and that there exists a positive function $\rho = \rho(r, T)$ on $(0, +\infty) \times (0, +\infty)$ satisfying the following condition (S_n) for each sufficiently large integer $n > 2T\omega(\rho(r, T))$:

(S_n) The system of equations:

$$x_\lambda^{(1)} - \lambda A x_\lambda^{(1)} \ni x, \quad x_\lambda^{(2)} - \lambda A x_\lambda^{(2)} \ni x_\lambda^{(1)}, \dots, \quad x_\lambda^{(n)} - \lambda A x_\lambda^{(n)} \ni x_\lambda^{(n-1)},$$

has a solution $\{x_\lambda^{(1)}, x_\lambda^{(2)}, \dots, x_\lambda^{(n)}\}$, where each $x_\lambda^{(v)}$ belongs to $B_{\rho(r, T)} \cap D(A)$, for every $x \in B_r \cap \overline{D(A)}$ and $\lambda \in (0, T/n]$.²⁾ Then

(2.1) $\exp(tA) \cdot x = \lim_{n \rightarrow \infty} \{I - (t/n)A|_{B_{\rho(r, T)}}\}^{-n} x, \quad 0 \leq t \leq T, \quad x \in B_r \cap \overline{D(A)},$
exists in X for each $r, T > 0$, and $\{\exp(tA); 0 \leq t < +\infty\}$ is a semi-group

1) For the notion of "multiple-valued" operator, we refer to Kato [5], §2.

2) Since $0 < \lambda < \omega(\rho(r, T))^{-1}$, one can write that $x_\lambda^{(1)} = (I - \lambda A|_{B_{\rho(r, T)}})^{-1} x$, $x_\lambda^{(2)} = (I - \lambda A|_{B_{\rho(r, T)}})^{-2} x, \dots, x_\lambda^{(n)} = (I - \lambda A|_{B_{\rho(r, T)}})^{-n} x$.

on $\overline{D(A)}$. Moreover the convergence in (2,1) is uniform in $t \in [0, T]$, $\exp(tA)$ is a locally Lipschitz continuous operator for each fixed $t \geq 0$:

$$\|\exp(tA) \cdot x - \exp(tA) \cdot y\| \leq \exp[\omega(\rho(r, T))t] \cdot \|x - y\|, \quad 0 \leq t \leq T, \\ x, y \in B_r \cap \overline{D(A)}, \text{ for each } T, r > 0,$$

and $\exp(tA) \cdot x$ is a locally Lipschitz continuous function of $t \geq 0$ for each fixed $x \in D(A)$.

Sketch of the proof. We set $T, r > 0, n \geq m \gg 2T\omega(\rho(r, T)), \mu \in (0, T/n], \lambda \in (0, T/m], \mu \leq \lambda$ and $x \in B_r \cap D(A)$. Then by the argument parallel to that used in the proof of Theorem I in Crandall-Liggett [2], we obtain immediately the following estimate:

$$\|(I - \mu A|_{B_{\rho(r, T)}})^{-n}x - (I - \lambda A|_{B_{\rho(r, T)}})^{-m}x\| \\ \leq \{[(n\mu - \lambda m)^2 + n\mu(\lambda - \mu)]^{1/2} \cdot \exp[2\omega(\rho(r, T))(n\mu + m\lambda)] \\ + [m\lambda(\lambda - \mu) + (m\lambda - n\mu)^2]^{1/2} \cdot \exp[4\omega(\rho(r, T))n\mu]\} \cdot \|Ax\|;$$

where $\|Ax\| = \inf\{\|y\|; y \in Ax\}$ (cf. (1.9) of [2]). This surely enables us to prove Theorem 1.

3. Examples. Let S be a uniform space and $(C(S), \|\cdot\|_\infty)$ be the Banach lattice of all real-valued bounded uniformly continuous functions on S normed with the supremum of the absolute value. We denote by $C(S)^+$ the cone of all non-negative elements of $C(S)$ and define the norm $\|\cdot\|$ in the Banach space $X = C(S) \times C(S)$ by the relation:

$$\|(u, v)\|^2 = \|u\|_\infty^2 + \|v\|_\infty^2, \quad (u, v) \in X.$$

Suppose we are given a generator A of a "sub-Markov semi-group"³⁾ (of contractions) on $C(S)$: A is a densely defined closed linear operator such that $R(I - \lambda A) = C(S)$ for any $\lambda > 0$ and

$$(3,1) \quad \sigma((f - c)^+, Af) \leq 0$$

holds for every $f \in D(A)$ and every non-negative constant function c on S where

$$(3,2) \quad \sigma(f, g) = \begin{cases} \lim_{\varepsilon \downarrow 0} \sup_{s \in S(f, \varepsilon)} g(s), & f \neq 0, f \geq 0, \\ 0, & f = 0; \end{cases}$$

$S(f, \varepsilon) = \{s \in S; |f(s)| > \|f\|_\infty - \varepsilon\}$ (see Sato [7]), in other words, A is "completely dispersive" in Kunita's terminology. We define two operators A_1 and A_2 in X by the relations:

$$(3,3) \quad \begin{aligned} D(A_i) &= (D(A) \cap C(S)^+) \times C(S)^+ \subset X, & i &= 1, 2; \\ A_1(u, v) &= (Au + uv, -uv), & (u, v) &\in D(A_1), \\ A_2(u, v) &= (Au - uv, uv), & (u, v) &\in D(A_2). \end{aligned}$$

Lemma 2. Both A_1 and A_2 are D -operators in $X = C(S) \times C(S)$ with $\omega(r) = 2r$ and satisfy the assumption in Theorem 1 with $\rho(r, T) = r \exp(2rT)$.

Proof. For a "semi-scalar product"⁴⁾ $[\cdot, \cdot]_\infty$ in $C(S)$, we define a semi-scalar product $[\cdot, \cdot]$ in $C(S) \times C(S)$ by the relation:

3) For this notion, we refer to Kunita [6].

4) For the definition, see, for example, Yosida [10].

$$[(u_1, v_1), (u_2, v_2)] = [u_1, u_2]_\infty + [v_1, v_2]_\infty, \quad (u_i, v_i) \in X, \quad i=1, 2.$$

Then we have easily that

$$[A_i(u_1, v_1) - A_i(u_2, v_2), (u_1, v_1) - (u_2, v_2)] \leq 2r \|(u_1, v_1) - (u_2, v_2)\|^2, \\ i=1, 2, \text{ for any } (u_j, v_j) \in B_r \cap D(A_i) \ (r > 0), \ j=1, 2. \text{ This implies that } A_1 \\ \text{ and } A_2 \text{ are D-operators with } \omega(r)=2r.$$

Next we shall prove (S_n) , $n > 2T\omega(\rho(r, T))$, for $A=A_1$ with $\rho(r, T) = r \exp(2rT)$. For each $(u, v) \in B_r \cap \overline{D(A_1)}$ and $\lambda \in (0, T/n]$, there exist solutions $u_\lambda^{(1)} \in D(A) \cap C(S)^+$, $v_\lambda^{(1)} \in C(S)^+$ of the equations: $u_\lambda^{(1)} - \lambda A u_\lambda^{(1)} - \lambda v u_\lambda^{(1)}(1 + \lambda u_\lambda^{(1)})^{-1} = u$, $v_\lambda^{(1)} = v(1 + \lambda u_\lambda^{(1)})^{-1}$, since $(I - \lambda A)^{-1}$ is non-negative and the operator $F_\lambda^{(1)}$ of $C(S)^+$ into itself defined by $F_\lambda^{(1)}f = vf(1 + \lambda f)^{-1}$, $f \in C(S)^+$, is a Lipschitz continuous operator with the Lipschitz constant $\leq \|v\|_\infty \leq r$. Moreover we have the estimates: $\|u_\lambda^{(1)}\|_\infty \leq (1 - \lambda \|v\|_\infty)^{-1}u$, $\|v_\lambda^{(1)}\|_\infty \leq \|v\|_\infty$. Accordingly the equation: $(u_\lambda^{(1)}, v_\lambda^{(1)}) - \lambda A_1(u_\lambda^{(1)}, v_\lambda^{(1)}) = (u, v)$, has a solution $(u_\lambda^{(1)}, v_\lambda^{(1)}) \in B_{r \exp(2rT)} \cap D(A_1)$. By the same argument the system of equations: $(u_\lambda^{(2)}, v_\lambda^{(2)}) - \lambda A_1(u_\lambda^{(2)}, v_\lambda^{(2)}) = (u_\lambda^{(1)}, v_\lambda^{(1)})$, $\dots$, $(u_\lambda^{(n)}, v_\lambda^{(n)}) - \lambda A_1(u_\lambda^{(n)}, v_\lambda^{(n)}) = (u_\lambda^{(n-1)}, v_\lambda^{(n-1)})$, has a solution $(u_\lambda^{(2)}, v_\lambda^{(2)}), \dots, (u_\lambda^{(n)}, v_\lambda^{(n)}) \in B_{r \exp(2rT)} \cap D(A_1)$, which satisfies the estimate: (3,4)

$$\|u_\lambda^{(n)}\|_\infty \leq (1 - \lambda \|v\|_\infty)^{-n} \|u\|_\infty, \quad \|v_\lambda^{(n)}\|_\infty \leq \|v\|_\infty.$$

Finally we shall prove (S_n) , $n > 2T\omega(\rho(r, T))$, for $A=A_2$ with $\rho(r, T) = r \exp(2rT)$. For each $(u, v) \in B_r \cap \overline{D(A_2)}$ and $\lambda \in (0, T/n]$, there exists a solution $u_\lambda^{(1)} \in D(A) \cap C(S)^+$ of the equation: $u_\lambda^{(1)} - \lambda A u_\lambda^{(1)} + \lambda v u_\lambda^{(1)}(1 - \lambda \min(r, u_\lambda^{(1)}))^{-1} = u$, since the operator $G_\lambda^{(1)}$ on $C(S)^+$ defined by $G_\lambda^{(1)}f = -vf(1 - \lambda \min(r, f))^{-1}$ is dissipative⁵⁾ and locally Lipschitz continuous and, accordingly, one can use Theorem 1.2.2 in Da Prato [3]. In view of (3,1) and (3,2) we have (see Sato [7])

$$\|(u_\lambda^{(1)} - \|u\|_\infty)^+\|_\infty = \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, u_\lambda^{(1)} - \|u\|_\infty) = \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, \\ u + \lambda A u_\lambda^{(1)} - \lambda v u_\lambda^{(1)}(1 - \lambda \min(r, u_\lambda^{(1)}))^{-1} - \|u\|_\infty) \leq \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, u - \|u\|_\infty) \\ + \lambda \sigma((u_\lambda^{(1)} - \|u\|_\infty)^+, -v u_\lambda^{(1)}(1 - \lambda \min(r, u_\lambda^{(1)}))^{-1}) \leq \|(u - \|u\|_\infty)^+\|_\infty = 0$$

(see also Sato [8]). Hence $u_\lambda^{(1)} \leq \|u\|_\infty \leq r$. Accordingly $(u_\lambda^{(1)}, v_\lambda^{(1)})$, $v_\lambda^{(1)} = v(1 - \lambda u_\lambda^{(1)})^{-1}$, satisfies $(u_\lambda^{(1)}, v_\lambda^{(1)}) - \lambda A_2(u_\lambda^{(1)}, v_\lambda^{(1)}) = (u, v)$, $(u_\lambda^{(1)}, v_\lambda^{(1)}) \in B_{r \exp(2rT)} \cap D(A_2)$. By the same argument the system of equations: $(u_\lambda^{(2)}, v_\lambda^{(2)}) - \lambda A_2(u_\lambda^{(2)}, v_\lambda^{(2)}) = (u_\lambda^{(1)}, v_\lambda^{(1)})$, $\dots$, $(u_\lambda^{(n)}, v_\lambda^{(n)}) - \lambda A_2(u_\lambda^{(n)}, v_\lambda^{(n)}) = (u_\lambda^{(n-1)}, v_\lambda^{(n-1)})$, has a solution $(u_\lambda^{(2)}, v_\lambda^{(2)}), \dots, (u_\lambda^{(n)}, v_\lambda^{(n)}) \in B_{r \exp(2rT)} \cap D(A_2)$, which satisfies the estimate:

$$(3,5) \quad \|u_\lambda^{(n)}\|_\infty \leq \|u\|_\infty, \quad \|v_\lambda^{(n)}\|_\infty \leq (1 - \lambda \|u\|_\infty)^{-n} \|v\|_\infty. \quad \text{Q.E.D.}$$

Applying Theorem 1 to the operators A_1 and A_2 , we have by the above-mentioned Lemma 2 and in view of the estimates (3,4) and (3,5):

The operators A_1 and A_2 defined by (3,3) generate semi-groups $\{\exp(tA_i); 0 \leq t < +\infty\}$, $i=1, 2$, respectively on the cone $C(S)^+ \times C(S)^+$

5) This follows from the fact: $\tau(f - g, G_\lambda^{(1)}f - G_\lambda^{(1)}g) \leq 0$ whenever $f, g \in C(S)^+$; where $\tau(f, g) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(\|f + \varepsilon g\|_\infty - \|f\|_\infty) = \limsup_{\varepsilon \downarrow 0} (\operatorname{sgn} f(s))g(s)$, if $f \neq 0$, $= \|g\|_\infty$, if $f = 0$ (cf. Hasegawa [4] and Sato [7]).

of $X = C(S) \times C(S)$ in the sense of Theorem 1. Moreover we have the following estimates: Set $(u_1(t), v_1(t)) = \exp(tA_1) \cdot (u, v)$ and $(u_2(t), v_2(t)) = \exp(tA_2) \cdot (u, v)$ for $u, v \in C(S)^+$ and $t \geq 0$, then

$$\begin{aligned} \|u_1(t)\|_\infty &\leq \exp(t\|v\|_\infty) \cdot \|u\|_\infty, & \|v_1(t)\|_\infty &\leq \|v\|_\infty; \\ \|u_2(t)\|_\infty &\leq \|u\|_\infty, & \|v_2(t)\|_\infty &\leq \exp(t\|u\|_\infty) \cdot \|v\|_\infty. \end{aligned}$$

Remark. The forms of operators A_1 and A_2 are suggested by the forth and fifth examples of weakly coupled diffusion systems in Yamaguti, Kametaka and Mimura [9]:

$$\begin{cases} \partial u / \partial t = Au + uv \\ \partial v / \partial t = -uv, \end{cases} \quad \begin{cases} \partial u / \partial t = Au - uv \\ \partial v / \partial t = uv. \end{cases}$$

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