3. On Integral Inequalities Related with a Certain Nonlinear Differential Equation

By Tominosuke OTSUKI
Tokyo Institute of Technology
(Comm. by Kinjirō KUNUGI, M. J. A., Jan. 12, 1972)

As is shown in [3], the following nonlinear differential equation:

\[ nh(1-h^2) \frac{d^2h}{dt^2} + \left( \frac{dh}{dt} \right)^2 + (1-h^2)(nh^2-1) = 0, \]

where \( n \) is any integer \( \geq 2 \), is the equation for the support function \( h(t) \) of a plane curve in the unit disk: \( u^2 + v^2 < 1 \), with respect to the tangent direction angle \( t \), which is related with a minimal hypersurface in the \((n+1)\)-dimensional unit sphere. Any solution \( h(t) \) of (1) such that \( h^2 + \left( \frac{dh}{dt} \right)^2 < 1 \) is periodic and its period \( T \) is given by the improper integral:

\[ T(C) = \int_{a_0}^{a_1} \frac{dh}{\sqrt{1-h^2-C \left( \frac{1}{h^2} - 1 \right)}} \]

where \( C = (a_0^2)^{1/n}(1-a_0^2)^{1-(1/n)} = (a_1^2)^{1/n}(1-a_1^2)^{1-(1/n)} \)

\[ 0 < a_0 < \frac{1}{\sqrt{n}} < a_1 \]

is the integral constant of (1). Regarding the function \( T(C) \), \( 0 < C < A = (1/n)^{1/n}(1-(1/n))^{1-(1/n)} \), the following is known in [3]:

(i) \( T(C) \) is differentiable and \( T(C) > \pi \),

(ii) \( \lim_{C \to 0} T(C) = \pi \) and \( \lim_{C \to A} T(C) = \sqrt{2} \pi \).

Putting \( h^2 = x \), \( a_0^2 = x_0 \), \( a_1^2 = x \), and \( 1/n = \alpha \), (2) can be written as

\[ T(C) = \int_{x_0}^{x_1} \frac{dx}{\sqrt{x(1-x) - C \psi(1-x)}}, \]

where

\[ \psi(x) = x^{\alpha}(1-x)^{1-\alpha} \quad \text{on} \quad 0 < x < 1 \]

and

\[ C = \psi(x_0) = \psi(x_1), \quad 0 < x_0 < \alpha < x_1 < 1, \]

\( 0 < C < A = \psi(\alpha) \).

Now, suppose that \( \alpha \) is any real number such that

\[ 0 < \alpha \leq 1/2 \]

and consider as the function \( T(C) \) is defined by the right hand side of (3) on the interval (6). Then, we have

* Dedicated to Professor Yoshie Katsurada on her 60th birth day.
Theorem. For the integral $T(C)$, we have the following inequality:

$$T(C) < \left( \frac{1}{\sqrt{2}} + \sqrt{1-\alpha} \right) \pi.$$  

Proof. We have easily

$$(8) \quad \psi(x)\psi(1-x) = x(1-x),$$

$$(9) \quad \frac{d\psi(x)}{dx} = \frac{1-x}{x(1-x)} \psi(x)$$

and

$$(10) \quad \frac{d\psi(1-x)}{dx} = \frac{1-x}{x(1-x)} \psi(1-x).$$

\(\psi(x)\) is monotone increasing on \(0 < x < \alpha\) and monotone decreasing on \(\alpha < x < 1\). Let \(X_0(u)\) and \(X_1(u)\) be the inverse functions of \(u = \psi(x)\) on \(0 < x < \alpha\) and \(\alpha < x < 1\) respectively. Thus, changing the integral parameter \(x\) in (3) to \(u = \psi(x)\) and using (8) and (9), \(T(C)\) can be written as

$$T(C) = \int_a^b dx \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}} + \int_0^1 dx \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}}$$

$$= \int_a^c \frac{\sqrt{X_0(u)}(1-X_0(u))(A-u)}{(\alpha-X_0(u))\sqrt{u(u-C)}} du$$

$$+ \int_c^d \frac{\sqrt{X_1(u)}(1-X_1(u))(A-u)}{(\alpha-X_1(u))\sqrt{u(u-C)}} du$$

$$= \int_a^c \frac{\sqrt{X_0(u)}(1-X_0(u))(A-u)}{(\alpha-X_0(u))\sqrt{u(u-C)}} du$$

$$+ \int_c^d \frac{\sqrt{X_1(u)}(1-X_1(u))(A-u)}{(\alpha-X_1(u))\sqrt{u(u-C)}} du$$

Now, we assume that

$$(11) \quad \frac{\sqrt{X_i(u)}(1-X_i(u))(A-u)}{\sqrt{u(u-C)}} \leq \lambda_i$$

for \(C \leq u < A\), \(i=0, 1\). Then, we have

$$(12) \quad T(C) < (\lambda_0 + \lambda_1) \int_a^c \frac{du}{\sqrt{(A-u)(u-C)}} = (\lambda_0 + \lambda_1) \pi.$$  

In the following, we shall show that we can take the values of \(\lambda_0\) and \(\lambda_1\) as

$$\lambda_0 = 1/\sqrt{2} \quad \text{and} \quad \lambda_1 = \sqrt{1-\alpha}.$$  

The inequalities (11) are equivalent to

$$(13) \quad \frac{\sqrt{x(1-x)}(A-\psi(x))}{|\alpha-x\sqrt{\psi(x)}} \leq \lambda_i$$

for \(x_0 \leq x < \alpha\) and \(\alpha < x \leq x_1\) respectively. Setting \(\lambda = \lambda_0\), \(\lambda_1\), (13) is equivalent to

$$x(1-x)(A-\psi(x)) \leq \lambda^2(\alpha-x)\psi(x),$$

that is

$$x(1-x)A \leq \psi(x)[\lambda^2(\alpha-x)^2 + x(1-x)].$$

By (8), this inequality can be written as
(14) \[ A \leq \frac{\lambda^2(\alpha - x)^2 + x(1-x)}{\psi(1-x)} := f_2(x). \]

For this positive valued function \( f_2(x) \) on \( 0 < x < 1 \) for any \( \lambda > 0 \), we have

(15) \[ f_2(\alpha) = A, \]

and

\[
\frac{f_i'}{f_i} = -2\lambda^2(\alpha - x) + \frac{1 - 2x}{\lambda^2(\alpha - x)^2 + x(1-x)} \frac{1 - \alpha - x}{x(1-x)}
\]

\[
= \frac{g_i(x)}{x(1-x)(\lambda^2(\alpha - x)^2 + x(1-x))},
\]

where

(16) \[ g_i(x) = (\alpha - x)[ -\lambda^2(1 - \alpha) + (1 - \lambda^2)x(1-x)]. \]

i) Case \( \lambda = \frac{1}{\sqrt{2}} \). We have

\[ g_i(x) = \frac{1}{2}(x - \alpha)^2(1 - \alpha - x) \]

which shows that (14) holds on the interval \( 0 < x < \alpha \), but not on any interval \( (\alpha, x_i) \).

ii) Case \( \lambda = \sqrt{1 - \alpha} \). We have

\[ g_i(x) = (\alpha - x)[ -\lambda^2(1 - \alpha) + \alpha x(1-x)]\]

\[ = \alpha(x - \alpha)[x^2 - x + (1 - \alpha)^2] \]

and

\[ 1 - 4(1 - \alpha)^2 \leq 1 - 4\left(1 - \frac{1}{2}\right)^2 = 0 \]

by (7), which shows that (14) holds on the interval \( 0 < x < 1 \).

Thus, we have proved that (11) are true when we put \( \lambda_i = 1/\sqrt{2} \) and \( \lambda_i = \sqrt{1 - \alpha} \). Hence, we get from (12)

\[ T(C) < \left(\frac{1}{\sqrt{2}} + \sqrt{1 - \alpha}\right)\pi. \]

Q.E.D.

Remark. The author wanted originally to have the inequality: \( T(C) < 2\pi \) from the standpoint of a geometrical problem and S. Furuya gave firstly an answer to it by proving the inequality: \( T(C) < \sqrt{(n-1)/n} \times 2\pi \) in [1]. By means of a numerical analysis and observation on (1) done by M. Urabe, it is expected to have the inequality: \( T(C) < \sqrt{2} \times \pi \) in [4].

References

