108. On Exponential Semigroups. II

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1. Introduction. Tamura and Shafer proved in [3] the following:

Theorem 1. If $S$ is an exponential archimedean semigroup with idempotent, then $S$ is an ideal extension of $I$ by $N$ where $I$ is the direct product of an abelian group $G$ and a rectangular band $B$ and $N$ is an exponential nil-semigroup.

However, the converse is not necessarily true. For example, let $S = \{a, b, c, d\}$ be the semigroup of order 4 defined by $(x, y \in S) \ xy = y$ for $y \neq d$ and all $x$; $xd = a$ for $x \neq c$; $cd = b$.

$S$ is the ideal extension of a right zero semigroup $\{a, b, c\}$ by a null semigroup of order 2. Associativity of $S$ is easily verified, but $S$ is not exponential:

$$(cd)^2 = b^2 = b, \quad c^3 d^3 = ca = a.$$ 

The purpose of this paper is to prove Theorem 2 which characterizes exponential ideal extensions of $I$ by $N$, and to give an alternate proof of the fact that $I$ is completely simple. See the definition of the used terminology in [3] and [1]. The notation may be different from that in [1].

Theorem 2. $S$ is an exponential archimedean semigroup with idempotent if and only if $S$ is an ideal extension of the direct product $I = A \times G \times M$ of a left zero semigroup $A$, an abelian group $G$, and a right zero semigroup $M$ by an exponential nil-semigroup $N$, with product determined by three partial homomorphisms $\phi : N \setminus \{0\} \rightarrow M$, $\otimes : N \setminus \{0\} \rightarrow G$, $\psi : N \setminus \{0\} \rightarrow A$ in the following manner. Let $(\lambda, a, \mu), (\nu, b, \eta) \in A \times G \times M$, $s, t \in N \setminus \{0\}$.

$$(\lambda, a, \mu) \cdot s = (\lambda, a(s \otimes), s \phi) \quad s \cdot (\lambda, a, \mu) = (a, s \otimes a, \mu)$$

$$(\lambda, a, \mu) \cdot (\nu, b, \eta) = (\lambda, ab, \eta) \quad s \cdot t = \begin{cases} st & \text{if } st \neq 0 \text{ in } N \\ (\psi s, (s \otimes)(t \otimes), t \phi) & \text{if } st = 0 \text{ in } N \end{cases}$$

2. Alternate proof of complete simplicity of $I$. In [3] Anderson's theorem on bicyclic subsemigroup was used, but we will derive primitiveness of idempotent elements. Assume that $S$ is an exponential archimedean semigroup. Let $e$ be an idempotent element of $S$ and let $I = SeS$. Since $I \subseteq SaS$ for all $a \in S$, $I$ is the kernel of $S$ and hence $I$ is simple. Let $e$ and $f$ be idempotents such that $ef = fe = f$. Now $IeI$
there exist \( x', y' \in I \) such that \( x'f y' = e \). Let \( x = ex'f \) and \( y = fy'e \). Then \( xy = (ex'f)(fy'e) = e(x'f y')e = e \). Since \( y = ye, yx = (ye)x = y(xy)x = (yx)^2 = y^2x^2 \) by exponentiality while \( xy = e \) implies \( e = (xy)(xy) = x(yx)y = e(yx)yx = eyx \) as we have \( ey = y \) and \( xe = x \). Finally \( f = ef = (yx)f = y(ex'f) = yx = e \).

Hence \( I \) is completely simple.

3. Preliminaries on ideal extension. Let \( D \) be a completely simple semigroup and let \( D = \mathcal{H}(A, G, M; F) \) be the Rees regular matrix representation of \( D \) where \( A \) is a left zero semigroup, \( G \) a group, \( M \) a right zero semigroup and \( F \) a sandwich matrix. Each element of \( D \) is expressed as

\[(\lambda, x, \mu), \lambda \in A, x \in G, \mu \in M.\]

The following are already known in [1], [2] or will be easily proved by readers.

(3.1) Let \( h : M \to M \) and \( p : M \to G \) be mappings. If we define \( \varphi(p, h) : D \to D \) by

\[\varphi(p, h)(\lambda, x, \mu) = (\lambda, x(p\mu), \mu h)\]

then \( \varphi(p, h) \) is a right translation of \( D \). Every right translation of \( D \) is obtained in this manner, and the correspondence \((p, h) \to \varphi(p, h)\) is one to one.

(3.2) Let \( k : A \to A \) and \( q : A \to G \) be mappings. If we define \( \psi((k, q)) : D \to D \) by

\[\psi((k, q))(\lambda, x, \mu) = (k\lambda, (q\mu)x, \mu)\]

then \( \psi((k, q)) \) is a left translation of \( D \) and every left translation of \( D \) is obtained in this manner. The correspondence \((k, q) \to \psi((k, q))\) is one to one.

(3.3) Let \( F = (f_{\mu, \lambda}) \), \( \mu \in M, \lambda \in A \). Then \( \varphi(p, h) \) is linked with \( \psi((k, q)) \) if and only if

\[(\mu \lambda) \cdot f_{\mu, \lambda} = f_{\mu, k\lambda} \cdot (q\lambda) \quad \text{for all} \quad \mu \in M, \lambda \in A.\]

In the present paper we deal with \( I = A \times G \times M \) (for \( D \)) in which all \( f_{\mu, \lambda} \) equal to the identity \( e \) of \( G \). Hence we have

\[(3.4) \mu \lambda = q\lambda \quad \text{for all} \quad \mu \in M, \lambda \in A.\]

Thus \( p \) and \( q \) are constant mappings taking the same value in \( G \). The \( p \) and \( q \) are denoted by \( p_a \) and \( q_a \) respectively, that is, \( \mu p_a = a, q_a \lambda = a \) for all \( \mu \in M, a \in A. \)

\[\psi((k_1, q_1)) \cdot \psi((k_2, q_2)) = \psi((k_1 k_2, q_1 q_2)), \]
\[\varphi(p_a, h_1) \cdot \varphi(p_a, h_2) = \varphi(p_a, h_1 h_2).\]

The translational hull \( \mathcal{H}(I) \) of \( I \) consists of \((\psi((k, q)), \varphi(p, h))\) and

\[
(\psi((k_1, q_1)), \varphi(p, h_1)) (\psi((k_2, q_2)), \varphi(p, h_2)) = (\psi((k_1 k_2, q_1 q_2)), \varphi(p, h_1 h_2)).
\]

(3.5) \( \varphi(p_a, h) (\psi((k, q))) \) is an inner right (left) translation of \( I \) if and only
if \( h(k) \) is a constant mapping. We redenote \( h(k) \) by \( h_{\mu_k}(k_{\mu_k}) \), i.e., \( \mu_{h_{\mu_k}} = \mu_k \) for all \( \mu \in M \), \( (k_{\lambda}, \lambda = \lambda_k \) for all \( \lambda \in \Lambda \). Then \((\lambda, x, \mu)\psi_{(\mu_{\mu_k}, k_{\mu_k})} = (\lambda, x, \mu)\) \((\zeta, \alpha, \mu_\zeta)\) for all \( \zeta \in \Lambda \).

\[
\psi_{(k_{\mu_k}, a_\mu)}(\lambda, x, \mu) = (\lambda, \alpha, \eta)(\lambda, x, \mu) \quad \text{for all } \eta \in M.
\]

The translational hull \( \mathcal{H}(I) \) is isomorphic onto the direct product \( \mathcal{I}_A \times \mathcal{I}_M = \{[k, a, h] : k \in \mathcal{I}_A, a \in G, h \in \mathcal{I}_M\} \) where \( \mathcal{I}_A \) and \( \mathcal{I}_M \) are the full-transformation semigroups on \( A \) and \( M \) respectively, under the map

\[
(\psi_{(k_{\mu_k}, a_\mu)}(\lambda, x, \mu))_{\mathcal{H}} \mapsto [k, a, h].
\]

Let \( \mathcal{M}(I) = \{(\psi_{(k_{\mu_k}, a_\mu)}(\lambda, x, \mu)) : \lambda \in \Lambda, a \in G, \mu \in M\} \). Since \( I \) is weakly reductive, \( \mathcal{M}(I) \) is isomorphic onto \( I \) under the composition:

\[
(\psi_{(k_{\mu_k}, a_\mu)}(\lambda, x, \mu))_{\mathcal{H}} \mapsto [k, a, h]_{\mathcal{H}}.
\]

After identifying \( [k, a, h]_{\mathcal{H}} \), let \( (\psi_{(k_{\mu_k}, a_\mu)}(\lambda, x, \mu)) = [k, a, h] \).

### 4. Exponential ideal extension.

Since \( I \) is weakly reductive, an ideal extension of \( I \) by \( N \) is determined by a partial homomorphism \( P^* \) of \( N^* = N \setminus \{0\} \) into \( \mathcal{H}(I) \) which satisfies

\[
P^*(s)P^*(t) \in \mathcal{M}(I) \quad \text{if } s, t \in N^* \text{ and } st = 0 \text{ in } N
\]

(See [1], [2]). For the notational convenience \( P^*(s) \) is denoted by

\[
P^*(s) = [k^{(s)}, g^{(s)}, h^{(s)}] \quad \text{where } k^{(s)} \in \mathcal{I}_A, g^{(s)} \in G \quad \text{and} \quad h^{(s)} \in \mathcal{I}_M.
\]

Now extend \( P^* \) to \( P \) on \( S = I \cup N^* \) as follows

\[
\begin{align*}
P(s) &= P^*(s) & \text{if } s \in N^* \\
P(\lambda, a, \mu) &= [k, a, h] & \text{if } (\lambda, a, \mu) \in I
\end{align*}
\]

where \( k, a, h \) are constant mappings. After identifying \( [k_{\mu_k}, a_{\mu_k}, h_{\mu_k}] \) with \( (\lambda_{\mu_k}, a_{\mu_k}, \mu_{\mu_k}) \), the operation on \( S \) can be expressed as follows:

\[
\begin{align*}
(\lambda, a, \mu)(\nu, b, \eta) &= (\lambda, ab, \eta) = P(\lambda, a, \mu)P(\nu, b, \eta) \\
(\lambda, a, \mu) \cdot s &= (\lambda, a, g^{(s)}a, s) = P(\lambda, a, \mu)P^*(s) \\
s(\lambda, a, \mu) &= (k^{(s)}, g^{(s)}a, \mu) = P^*(s)P(\lambda, a, \mu)
\end{align*}
\]

(4.2)

Accordingly

\[
\begin{align*}
xy &= P(x)P(y) = P(xy) & \text{if } xy \in I \\
P(xy) &= P(x)P(y) & \text{for all } x, y \in S.
\end{align*}
\]

Thus \( P \) is a homomorphism of \( S \) into \( \mathcal{M}(I) \).

Assume that we obtain an exponential ideal extension \( S \) of \( I \) by an exponential nil-semigroup \( N \). Let \( s \in N^* \). Since \( N \) is nil, there is a positive integer \( n \) such that \( s^n \in I \), hence \( (P(s))^n \in \mathcal{M}(I) \), i.e. \( k^{(s)^n} \) and \( h^{(s)^n} \) are constant mappings. Let \( (P(s))^n = [k^n, g^n, h^n] \).

By exponentiality of \( S \),

\[
(\lambda, a, \mu) \cdot s^n = (\lambda, a, \mu)^n s^n \quad \text{for all } (\lambda, a, \mu) \in I, s \in N^*.
\]

By (4.3) we get
On the other hand

\[ (4.5) \]

From the equality 

\( (4.4) = (4.5) \), we have

\[ h_\mu h^{(s)} = h_\mu, \quad \text{for all } \mu \in M, \]

that is,

\[ h^{(s)} = h_\mu \quad \text{for all } \mu \in M. \]

Hence \( h^{(s)} \) is a constant mapping. Similarly, starting \((s \cdot (\lambda, a, \mu))^{(s)} = s^{(\lambda, a, \mu)}\), we can prove that \( k^{(s)} \) is a constant mapping.

Consequently \( P^* \) induces mappings

\[ \psi : N^* \to A, \quad \xi : N^* \to G, \quad \varphi : N^* \to M \]

such that \( P^*(s) = [k_{s^r}, s_{(s)}, h_{s}] \). \( P^* \) is a partial homomorphism of \( N^* \) into \( A(I) \), and hence \( P \) is a homomorphism \( S \) into \( A(I) \). Thus we have obtained (2.1).

Conversely assume that \( \psi, \xi, \varphi \) are given and that \( S \) is defined by (2.1). The three mappings induce a partial homomorphism \( P^* \) of \( N^* \), \( P^*(s) = [k_{s^r}, s_{(s)}, h_{s}] \), and hence induces a homomorphism \( P \) of \( S \) into \( A(I) \) by (4.1). Associativity of \( S \) is assured by the general theory of ideal extension of a weakly reductive semigroup, and so we need only to show exponentiality of \( S \):

\[ (xy)^m = x^my^m \quad \text{for all } x, y \in S, \quad \text{for } m > 1. \]

First note that \( I \) is medial; hence \( P(S) \) is medial.

If \( x^my^m \in I \) then (4.6) is obtained by the exponentiality of \( N \). If \( x^my^m \in I \), then \((xy)^m \in I \) and, by (4.3) and the above remark,

\[ (xy)y^m = P(x^m)(P(y^m) = (P(x))^m(P(y))^m, \]

\[ (xy)^m = P(xy)[P((xy)^m-1)] = P(xy)(P(x)P(y)(P(x)P(y))^{m-1} = P(x)P(y)(P(x)P(y))^{m-1} = P(x)^m(P(y))^{m-1} = P(x)(P(x))^{m-1}P(y)(P(y))^{m-1} = (P(x))^m(P(y))^m. \]

Hence (4.6) has been proved.

An ideal extension of \( I \) by \( N \) determined by a partial homomorphism \( N^* \to A(I) \) is called a strict ideal extension.

Thus we have Theorem 2' which is a restatement of Theorem 2 and also describes the "medial" case. The medial case is an immediate consequence from the fact that \( P(S) \) is medial.

Theorem 2'. \( S \) is an exponential (medial) archimedean semigroup with idempotent if and only if \( S \) is a strict ideal extension of the direct product of an abelian group \( G \) and a rectangular band \( B \) by an exponential (medial) nil-semigroup \( N \).

Finally we exhibit an example of exponential semigroup which is not medial. It is sufficient to show such a nil-semigroup. Let \( F \) be the free semigroup generated by two letters \( a, b \) and let \( S^* \) be a subset of \( F \) defined by
\[ S^* = \{a, b, ab, a^3, ba, a^2b, aba, a^2ba \} \]

and

\[ I = F \setminus S^*. \]

Then \( I \) is an ideal of \( F \). Let \( S = F/I \). \( S \) is an exponential semigroup of order 9 which is not medial since \( a^2ba \neq aba^3 = 0 \).

References

