122. Remarks on the Asymptotic Behavior of the Solutions of Certain Non-Autonomous Differential Equations

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1. Introduction. In this paper we consider the asymptotic behavior of the solutions of the non-autonomous, nonlinear differential equation;

$$(1.1) \quad \dot{x} = A(t)x + f(t, x)$$

where x, f are n -dimensional vectors, $A(t)$ is a bounded continuously differentiable $n \times n$ matrix for $t \geq 0$, and $f(t, x)$ is a continuous in (t, x) for $t \geq 0$, $\|x\| < \infty$, here $\|\cdot\|$ denotes an Euclidean norm. And consider

$$(1.2) \quad \ddot{x} + a(t)\dot{x} + b(t)g(x, \dot{x})\dot{x} + c(t)h(x) = p(t, x, \dot{x}, \ddot{x})$$

where $a(t), b(t), c(t)$ are positive, continuously differentiable and g, h, p are continuous real-valued functions depending only on the arguments shown, the dots indicate the differentiation with respect to t . In this note, certain conditions are obtained under which all solutions of (1.1) tend to zero as $t \rightarrow \infty$.

In [6], the author studied the asymptotic behavior of the solution of the equation

$$(1.3) \quad \ddot{x} + a(t)f(x, \dot{x})\dot{x} + b(t)g(x, \dot{x})\dot{x} + c(t)h(x) = e(t)$$

under the assumptions that $|a'(t)|, |b'(t)|, |c'(t)|$ and $e(t)$ are integrable and suitable conditions on f, g, h . Here we assume the condition that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \{|a'(s)| + |b'(s)| + |c'(s)|\} ds$$

has an infinitesimal upper bound,

to prove the every solution of (1.2) tends to zero as $t \rightarrow \infty$. Conditions on $p(t, x, y, z)$ are also relaxed. Theorem 2 generalizes the Ezeilo's result [5] in which he considered the equation

$$(1.4) \quad \ddot{x} + a_1\dot{x} + a_2\dot{x} + f_3(x) = p_1(t, x, \dot{x}, \ddot{x}),$$

where a_1, a_2 are positive constants.

The main tool used in this work is Lemma 1 which is a specialization of the result obtained by F. Brauer [1]. Using this Lemma and Liapunov functions, we shall obtain Theorem 1 and Theorem 2. Lemma 1 is especially convenient to study the non-autonomous differential equations.

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2. Main lemma. Consider a system of differential equations

$$(2.1) \quad \dot{x} = F(t, x),$$

where x and F are n -dimensional vectors.

Lemma 1. Suppose that $F(t, x)$ of (2.1) is continuous in $I \times R^n$ ($I = [0, \infty)$) and that there exists a Liapunov function $V(t, x)$ defined in $I \times R^n$ satisfying the following conditions;

(i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, where $a(r) \in CIP$ (i.e. continuous and increasing positive definite functions), $a(r) \rightarrow \infty$ as $r \rightarrow \infty$ and $b(r) \in CIP$,

(ii) $\dot{V}_{(2.1)}(t, x) \leq -cV(t, x) + \lambda_1(t)V(t, x) + \lambda_2(t)\phi(V(t, x))$, where $c > 0$ is a constant and $\lambda_i(t) \geq 0$ ($i = 1, 2$) are continuous functions satisfying

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda_1(s) ds < c, \quad \int_t^{t+1} \lambda_2(s) ds \rightarrow 0 \text{ as } t \rightarrow \infty,$$

and $\phi(u)$ is a continuous, non-negative function for $u \geq 0$ satisfying $\phi(u) = 0$ as $u \rightarrow \infty$.

Then, all solutions $x(t)$ of (2.1) are uniform-bounded and satisfy $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

The detailed proof of Lemma 1 is to appear in some journal.

3. Theorems. Let $A(t)$ satisfy the condition (i) of the following Theorem 1, and $P(t)$ be a solution of the matrix equation

$$(3.1) \quad A^T(t)P(t) + P(t)A(t) = -I.$$

Notice that $P(t)$ is bounded for bounded $A(t)$. The following propositions are due to J. R. Dickerson [2].

Proposition A. $x^T P(t)x \geq C\|x\|^2$, where C is a positive constant.

Proposition B. $|x^T \dot{P}(t)x| \leq 2\|\dot{A}(t)\| \cdot \|P(t)\| x^T P(t)x$, where $\dot{P}(t)$ and $\dot{A}(t)$ denote the time derivative of matrices $P(t)$ and $A(t)$ respectively.

Theorem 1. Suppose that the following conditions are satisfied;

(i) the eigenvalues of $A(t)$ have negative real parts strictly bounded away from zero for all $t \geq 0$,

(ii) $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|\dot{A}(s)\| ds < \frac{1}{2P_1^2}$

where $P_1 = \sup_{t \geq 0} \|P(t)\|$,

(iii) $\|f(t, x)\| \leq \gamma_1(t) + \gamma_2(t)\|x\|^\rho$

where $\gamma_1(t), \gamma_2(t)$ are non-negative, continuous for $t \geq 0$ and ρ is a constant such that $0 \leq \rho \leq 1$,

(iv) $\int_t^{t+1} \gamma_i(s) ds \rightarrow 0$ as $t \rightarrow \infty$ ($i = 1, 2$).

Then, all solutions $x(t)$ of (1.1) are uniform-bounded and satisfy $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Next, we consider the equation (1.2) and assume that $g(x, y)$ and $g_x(x, y)$ are continuous, real-valued for all (x, y) and $h(x)$ is continuously differentiable for all x .

Theorem 2. Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable functions, and the following conditions are satisfied;

$$(i) \quad A \geq a(t) \geq a_0 > 0, B \geq b(t) \geq b_0 > 0, C \geq c(t) \geq c_0 > 0, \\ \text{for } t \in I = [0, \infty),$$

$$(ii) \quad h(0) = 0, \frac{h(x)}{x} \geq \delta > 0 \quad (x \neq 0),$$

$$(iii) \quad 0 < g_0 \leq g(x, y) \leq g_0 + \frac{4\delta c_0}{Bb_0g_0}, \quad yg_x(x, y) \leq 0 \quad \text{for all } (x, y) \in R^2,$$

$$(iv) \quad \frac{a_0b_0g_0}{C} > h_1 \geq h'(x),$$

$$(v) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \{|a'(s)| + |b'(s)| + |c'(s)|\} ds$$

has an infinitesimal upper bound,

$$(vi) \quad |p(t, x, y, z)| \leq p_1(t) + p_2(t)(x^2 + y^2 + z^2)^{\rho/2} + A_1(x^2 + y^2 + z^2)^{1/2} \\ \text{where } \rho, A_1 \text{ are constants such that } 0 \leq \rho \leq 1, A_1 \geq 0 \text{ and } p_1(t), p_2(t) \\ \text{are non-negative, continuous functions,}$$

$$(vii) \quad \int_t^{t+1} p_i(s) ds \rightarrow 0 \text{ as } t \rightarrow \infty \quad (i=1, 2).$$

If A_1 is sufficiently small, then every solution $x(t)$ of (1.2) is uniform-bounded and satisfies $x(t) \rightarrow 0, \dot{x}(t) \rightarrow 0, \ddot{x}(t) \rightarrow 0$ as $t \rightarrow \infty$.

4. Proof of Theorems. For the proof of Theorem 1, we consider the Liapunov function

$$(4.1) \quad U(t, x) = x^T P(t) x.$$

By virtue of Proposition A and the boundedness of $P(t)$, we have

$$C \|x\|^2 \leq V(t, x) \leq P_1 \|x\|^2.$$

A simple calculation shows that

$$\dot{V}_{(1.1)}(t, x) \leq -\frac{1}{P_1} V(t, x) + 2P_1 \|A(t)\| V(t, x) \\ + 2\{\gamma_1(t) + \gamma_2(t)\} \left\{ \left(\frac{V(t, x)}{C} \right)^{1/2} + \left(\frac{V(t, x)}{C} \right)^{(1+\rho)/2} \right\}.$$

Hence, the assumptions of Lemma 1 hold and the proof of Theorem 1 is completed.

For the proof of Theorem 2, the following Liapunov function is constructed;

$$(4.2) \quad V(t, x, y, z) = V_0(t, x, y, z) + V_1(t, x, y, z)$$

where V_0 and V_1 are defined by

$$(4.3) \quad 2\mu_1 V_0 = 2\mu_1 c(t) \int_0^x h(\xi) d\xi + 2c(t)h(x)y + 2b(t) \int_0^y g(x, \eta) \eta d\eta \\ + \mu_1 a(t)y^2 + 2\mu_1 yz + z^2,$$

$$2V_1 = \mu_2 g_0 b(t)x^2 + 2a(t)c(t) \int_0^x h(\xi) d\xi + [a^2(t) - \mu_2]y^2$$

$$(4.4) \quad + 2b(t) \int_0^y g(x, \eta) \eta d\eta + z^2 + 2\mu_2 a(t)xy \\ + 2\mu_2 xz + 2a(t)yz + 2c(t)h(x)y$$

and

$$\frac{Ch_1}{b_0g_0} < \mu_1 < a_0, \quad 0 < \mu_2 < \frac{a_0b_0g_0 - Ch_1}{A}.$$

A good calculation shows that the above Liapunov function satisfies the hypotheses of Lemma 1.

The detailed proof of Theorem 2 is to appear in some journal with the proof of Lemma 1.

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