130. On Pseudoparacompactness and Continuous Mappings

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Throughout this paper we assume that spaces are completely regular T_1 -spaces and maps are continuous. The completion of a space X with respect to its finest uniformity is called the topological completion of X , and denoted by μX . According to Morita [8] a space X is called pseudoparacompact (resp. pseudo-Lindelöf) if μX is paracompact (resp. Lindelöf).

As for these notions, in the same paper Morita proved the following remarkable results.

Theorem 1 (Morita [8], Theorems 3.1, 3.2 and 3.5).

- (1) μX is compact iff X is pseudocompact.
- (2) μX is always a paracompact M -space for any M -space X .
- (3) Let X be an M -space. X is pseudo-Lindelöf iff it is the quasi-perfect inverse image of a separable metric space.

The characterizations of pseudoparacompactness and pseudo-Lindelöfness have been obtained by Howes [4] and Ishii [5] independently. On the other hand, in [2] Hanai and Okuyama (cf. Isiwata [6]) essentially proved the following result: "If a space X is the inverse image of a pseudocompact space under an open quasi-perfect map, then X is pseudocompact". Here the assumption that the map is open cannot be dropped in general ([3] Example 2.4). Analogously to this result, in § 1 we shall prove the following theorem which is a partial answer to a problem posed by Ishii [5] concerning (2) and (3) of Theorem 1: "Is pseudoparacompactness or pseudo-Lindelöfness preserved under taking the inverse image by a quasi-perfect (or perfect) map?"

Theorem 2. *If there is an open quasi-perfect map $\varphi: X \rightarrow Y$ from a space X onto a pseudoparacompact (resp. pseudo-Lindelöf) space Y , then X is pseudoparacompact (resp. pseudo-Lindelöf).*

In § 2, by virtue of recent results obtained by Morita, we shall prove the following

Theorem 3. *Let $\varphi: X \rightarrow Y$ be an open quasi-perfect map from a space X onto a space Y .*

- (1) *If μY is locally compact and paracompact, then so is μX .*
- (2) *If μY is σ -compact, then so is μX .*

§ 1. Proof of Theorem 2. Before proving Theorem 2, we shall

need some preliminaries. For a space X , let μ be the finest uniformity of X and ν the uniformity of all countable normal coverings of X .

Lemma 1.1 (Howes [4]). *A space X is pseudoparacompact (resp. pseudo-Lindelöf) iff for any weakly Cauchy filter $\mathfrak{F}$ with respect to μ (resp. ν) there exists a Cauchy filter $\mathfrak{G}$ with respect to μ containing $\mathfrak{F}$.*

Here a filter $\mathfrak{F}$ in X is called weakly Cauchy with respect to a uniformity μ of X if for any uniform cover $\mathfrak{U}$ in μ there is a filter $\mathfrak{G}$ in X containing $\mathfrak{F}$ such that $G \subset U$ holds for some $G \in \mathfrak{G}$ and $U \in \mathfrak{U}$.

Let $\mathcal{C}(X)$ be the family of all non-empty compact subsets of a given space X . Following the convention of [7], we topologize $\mathcal{C}(X)$ with the Vietoris topology; for a finite collection $\{U_1, U_2, \dots, U_n\}$ of open sets, $\langle U_1, U_2, \dots, U_n \rangle$ will denote the subset of $\mathcal{C}(X)$ to which the compact set K belongs iff $K \subset \bigcup_i U_i$ and $K \cap U_i \neq \emptyset$ for $i=1, 2, \dots, n$. Open sets in $\mathcal{C}(X)$ are unions of an arbitrary number of these sets.

Lemma 1.2 (Michael [7]). *$\mathcal{C}(X)$ is completely regular and T_1 iff X is completely regular and T_1 .*

A space X is called topologically complete if $\mu X = X$ (cf. [8]).

Lemma 1.3 (Zenor [10]). *$\mathcal{C}(X)$ is topologically complete iff X is topologically complete.*

A subset F of a space X is called relatively pseudocompact if every real-valued continuous function over X is bounded on F .

Lemma 1.4 (Dykes [1]). *If F is a relatively pseudocompact subset of a topologically complete space X , then $\text{cl}_X F$ is compact.*

A map $\varphi: X \rightarrow Y$ is called a Z -map if the image of each zero-set in X is closed in Y . In [6], Isiwata extended the notion of Z -maps; a map $\varphi: X \rightarrow Y$ is a WZ -map if $\text{cl}_{\mu X} \varphi^{-1}(y) = \beta(\varphi)^{-1}(y)$ for every y in Y , where $\beta(\varphi)$ denotes the Stone extension of φ .

The following lemma is useful.

Lemma 1.5. *Let $\varphi: X \rightarrow Y$ be a map from X onto Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each y in Y . For y in Y , let us put $\tilde{\varphi}(y) = \text{cl}_{\mu X} \varphi^{-1}(y)$. If φ is an open WZ -map, then the mapping $\tilde{\varphi}$ from Y into $\mathcal{C}(\mu X)$ is continuous. Conversely if $\tilde{\varphi}$ is continuous then φ is open, and moreover if X is normal then φ is closed.*

Proof. Clearly $\tilde{\varphi}$ maps Y into $\mathcal{C}(\mu X)$ by Lemma 1.4. Let φ be an open WZ -map, and for y in Y let $\tilde{\varphi}(y) \in \langle U_1, U_2, \dots, U_n \rangle$, where U_i is an open set in μX for each i . If we choose an open set U' in βX such that $U' \cap \mu X = \bigcup_i U_i$, then the set $V = \bigcap_i \varphi(U_i \cap X) \cap (\beta Y - \beta \varphi(\beta X - U'))$ is an open set in Y containing y since φ is an open WZ -map and $\text{cl}_{\mu X} \varphi^{-1}(y)$ is compact. Moreover we easily see $\tilde{\varphi}(V) \subset \langle U_1, U_2, \dots, U_n \rangle$. Therefore $\tilde{\varphi}$ is continuous. Conversely let us assume $\tilde{\varphi}$ is continuous. Let U be an open set in X and choose an open set U' in μX such that $U' \cap X = U$. Then $\varphi(U) = \tilde{\varphi}^{-1}(\langle U', \mu X \rangle \cap \tilde{\varphi}(Y))$. Hence φ is open. Now,

let us assume X is normal. For a closed set F in X , let us put $\mathfrak{F} = \{K \in \mathcal{C}(\mu X) \mid \text{cl}_{\mu X} F \cap K \neq \emptyset\}$. Then $\mathfrak{F}$ is closed in $\mathcal{C}(\mu X)$, and $\varphi(F) = \bar{\varphi}^{-1}(\mathfrak{F} \cap \bar{\varphi}(Y))$. Therefore φ is closed. This proves Lemma 1.5.

Theorem 2 is an immediate consequence of the following

Theorem 4. *Let $\varphi: X \rightarrow Y$ be an open WZ-map from a space X onto a pseudoparacompact (resp. pseudo-Lindelöf) space Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each y in Y , then X is pseudoparacompact (resp. pseudo-Lindelöf).*

Proof. Let $\mathfrak{F}$ be a weakly Cauchy filter in X with respect to μ (resp. ν). Then the filter $\varphi(\mathfrak{F})$ is weakly Cauchy with respect to μ (resp. ν) since φ is continuous. Moreover since Y is pseudoparacompact (resp. pseudo-Lindelöf), by Lemma 1.1 there exists a Cauchy filter $\mathfrak{G}$ in Y with respect to μ , which contains $\varphi(\mathfrak{F})$. Let $\bar{\varphi}$ be a map as in Lemma 1.5, then $\bar{\varphi}(\mathfrak{G})$ is also a Cauchy filter in $\mathcal{C}(\mu X)$ with respect to μ since $\bar{\varphi}$ is continuous. Therefore since $\mathcal{C}(\mu X)$ is topologically complete by Lemma 1.3, $\bar{\varphi}(\mathfrak{G})$ converges to some K in $\mathcal{C}(\mu X)$. Let us suppose that $\bigcap \{\text{cl}_{\mu X} F \mid F \in \mathfrak{F}\} \cap K = \emptyset$. Since K is compact, it follows that $\text{cl}_{\mu X} F \cap K = \emptyset$ for some $F \in \mathfrak{F}$. This means that $K \in \langle \mu X - \text{cl}_{\mu X} F \rangle$. Since $\bar{\varphi}(\mathfrak{G})$ converges to K , there exists G in $\mathfrak{G}$ such that $\bar{\varphi}(G) \subset \langle \mu X - \text{cl}_{\mu X} F \rangle$. Then it is easily seen that $\varphi^{-1}G \subset X - F$. But this contradicts that $\varphi(\mathfrak{F}) \subset \mathfrak{G}$. Hence $\mathfrak{F}$ has a cluster point in K . This shows that $\mathfrak{F}$ is contained in a Cauchy filter in X with respect to μ . Therefore X is pseudoparacompact (resp. pseudo-Lindelöf) by Lemma 1.1. The proof is completed.

Remark. Under the map $\varphi: X \rightarrow Y$ given in Theorem 4, let us assume that Y is pseudocompact and consider $\mathfrak{F}$ in the proof above to be a weakly Cauchy filter with respect to the uniformity of all finite normal coverings, then under the same argument as above, by ([4], Theorem 3) we can conclude that X is pseudocompact. This is an another proof of ([6], Theorem 4.2).

As an application of Theorem 4 we have

Theorem 5. *Let X be a pseudocompact space and Y a first countable and pseudoparacompact (resp. pseudo-Lindelöf) space. Then $X \times Y$ is pseudoparacompact (resp. pseudo-Lindelöf).*

Proof. Since the projection $X \times Y \rightarrow Y$ is a Z-map by ([6], Theorem 2.1), this follows from Theorem 4.

§ 2. Proof of Theorem 3. Theorem 3 is a direct consequence of the following lemma and theorems which are due to Morita.

Lemma 2.1. *Let $\varphi: X \rightarrow Y$ be an open WZ-map from X onto Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each y in Y . If F is a relatively pseudocompact subset of Y , then $\varphi^{-1}(F)$ is relatively pseudocompact.*

Proof. For any real-valued continuous function f on X , let us define real-valued functions f^s and f^i on Y by

$$f^s(y) = \sup \{f(x) \mid x \in \varphi^{-1}(y)\}, \quad f^i(y) = \inf \{f(x) \mid x \in \varphi^{-1}(y)\}.$$

Then f^s and f^i are continuous by ([6], Lemma 4.1) and bounded on F . Hence f is bounded on $\varphi^{-1}(F)$ and this proves Lemma 2.1.

Theorem 6 (Morita [9]). *For a space X , μX is locally compact and paracompact iff there exists a normal open covering of X consisting of relatively pseudocompact subsets.*

Theorem 7 (Morita). *For a space X , μX is σ -compact iff X is expressed as a union of a countable number of relatively pseudocompact subsets.*

Proof. Let $\mu X = \bigcup \{K_i \mid i=1, 2, \dots\}$, where each K_i is compact. Then $X = \bigcup_i (K_i \cap X)$ and since X is C -embedded in X by ([8], Theorem 2.4), $K_i \cap X$ is relatively pseudocompact. Conversely, suppose that $X = \bigcup \{F_i \mid i=1, 2, \dots\}$, where each F_i is relatively pseudocompact. Let us put $Y = \bigcup_i \text{cl}_{\mu X} F_i$. Then $X \subset Y \subset \mu X$ and Y is a σ -compact space by Lemma 1.4. Therefore by ([8], Theorem 2.5) it holds that $Y = \mu X$. Hence μX is σ -compact and this completes the proof of Theorem 7.

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