148. Iterated Loop Spaces

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(Comm. by Kenjiro Shoda, M. J. A., Nov. 13, 1972)

The aim of this note is to give conditions under which a space or a map can be de-looped \(k\)-times up to homotopy. The duals to Theorems 1 and 2 have been obtained by Berstein-Ganea [2]. Our basic lemma (Lemma 1) allows us to overcome the difficulty which arises in dualizing Theorem 3.3 of T. Ganea [4], thereby obtaining a de-looping theorem for a homotopy \(\Omega^kS^k\)-space (see Theorem 4).

1. A basic lemma. First we set up some notation and conventions. The spaces we consider are supposed to have the based homotopy type of \(CW\)-complexes. We denote the loop and suspension functors by \(\Omega\) and \(S\). Given a map \(u : A \to B\), the fibre \(\{(a, \gamma) \in A \times B^I ; \gamma(0) = *, \gamma(1) = u(a)\}\) and the cofibre \(B \cup_u CA\) are denoted by \(E_u\) and \(C_u\) respectively. The identity maps \(\Omega^kX \to \Omega^kX\) and \(S^kX \to S^kX\) yield the canonical adjointness maps \(e_k : S^k\Omega^kX \to X\) and \(\eta_k : X \to \Omega^kS^kX\).

Now given a map \(f : \Omega X \to Y\), introduce the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega X & \xrightarrow{f} & Y \\
\alpha' \downarrow & & \\
E_{i} & \xrightarrow{i} & C_{f} \\
\beta' \downarrow & \alpha \downarrow & \\
\Omega X & \xrightarrow{\beta} & E_{i} \\
\eta_k \downarrow & \beta \downarrow & \\
\Omega X & \xrightarrow{\eta_k} & \Omega C_{i} \\
\end{array}
\]

in which the vertical maps are constructed as in p. 132 of [6] using the canonical homotopies, \(i\) and \(j\) are inclusions and \(q : C_f \to S\Omega X\) the map pinching \(Y\) to a point. Using the Blakers-Massey theorem (see e.g. Theorem 4.3 of [8]) we have

i) \((\beta \alpha) f \simeq \Omega j\),

ii) the construction of \(\beta \alpha\) is functorial,

iii) if \(f\) is \(m\)-connected, \(m \geq 1\), \(X\) is \(2\)-connected and \(Y\) is \((n-1)\)-connected, \(n \geq 1\), then \(\beta \alpha\) is \([m + \min(m, n)]\)-connected, \(j(m+1)\)-connected and \(C_{i} = \min(n, 2m+1)\)-connected.

Iterating the process for \(j\), we get

**Lemma 1.** If \(f : \Omega^kX \to Y\) is \(m\)-connected such that \(X\) is \((k+1)\)-
connected and $Y$ is $(n-1)$-connected, $m \geq n > k-1 \geq 0$, then there exist an $(n+k-1)$-connected space $Z$ and an $(m+n)$-connected map $g: Y \to \Omega^k Z$ such that $gf$ is homotopic to a $k$-fold loop map. The construction of $g$ is functorial. Further, if $h: Y \to \Omega^k V$ with $V(k+1)$-connected is a map such that $hf$ can be de-looped $k$-times, then there exists a $\lambda: Z \to V$ with $(\Omega^k \lambda)g \simeq h$.

2. As an immediate consequence of Lemma 1 we obtain the following two theorems which are dual to Theorems 1.4 and 1.6 of [2], so the proofs are omitted.

**Theorem 1.** If $X$ is an $(n-1)$-connected space with $\pi_i(X) = 0$ for $i \geq 3n$, $n \geq 2$, such that $\eta$ has a homotopy retraction, then $X$ is homotopy equivalent to a $k$-fold loop space.

**Remark.** Taking $k=1$ in Theorem 1, we recover Theorem C of P. J. Hilton [5].

**Theorem 2.** Let $\phi: \Omega^k A \to \Omega^k B$ be a homotopy $\Omega^k S^k$-map, i.e. $(\Omega^k \varepsilon_\phi)(\Omega^k S^k \phi) \simeq \phi(\Omega^k \varepsilon_\phi)$. If $A$ is $(n-1)$-connected, $n > k + 1$, and $B$ is a 1-connected space with $\pi_i(B) = 0$ for $i \geq 3n - 2k + 1$, then $\phi$ is de-looped $k$-times.

The following theorem extends Theorem 5 of [7].

**Theorem 3.** Suppose $X$ and $Y$ are $n$- and $q$-connected respectively, $k + 2 \leq n \leq q - 2$, such that $\pi_i(X) = 0$ if $i \geq 2n + 2 + k$ and $\pi_j(Y) = 0$ if $j \geq q + n + 2 - k$. Then $f: \Omega^k X \to \Omega^k Y$ is homotopic to $k$-fold loop map provided that $E_f$ is of the same homotopy type as a $k$-fold loop space.

**Proof.** Denote by $p: \Omega^k E \to \Omega^k X$ the fibre of $f$. Since $p$ is $(q-k)$-connected and since $\Omega^k X$ is $(n-k)$-connected, it follows from Lemma 1 that there is an $(n+q-2k+1)$-connected map $\eta: \Omega^k X \to \Omega^k Z$ such that $gp \simeq \Omega^k j$ for some $j: E \to Z$. Moreover, since $fp \simeq 0$ is de-looped $k$-times, Lemma 1 gives a map $\lambda: Z \to Y$ with $(\Omega^k \lambda)g \simeq f$. Killing the homotopy of $Z$ in dimensions $\geq n + q - k + 2$, we get an $(n+q-k+2)$-connected inclusion $h: Z \subset W$, hence $h^*: [W, Y] \to [Z, Y]$ is onto. This gives rise to a map $\mu: W \to Y$ with $\mu h \simeq \lambda$. On the other hand, since $\varepsilon_\mu: S^k \Omega^k X \to X$ is $(2n+2-k)$-connected and since $\pi_i(Z) = 0$ for $2n + 2 - k \leq i \leq n + q - k + 1$, we see that $\varepsilon_\mu^*: [X, W] \to [S^k \Omega^k X, W]$ is onto, which yields a map $\nu: X \to W$ with $\nu \simeq \omega$ the adjoint of $(\Omega^k h)g$, whence $\Omega^k \nu \simeq (\Omega^k h)g$. Then $f \simeq \Omega^k (\mu)$ as desired.

3. Homotopy $\Omega^k S^k$-spaces. J. Beck [1] has shown that a $\Omega^k S^k$-space can always be de-looped $k$-times. We shall prove a theorem for a homotopy analogue (cf. Corollary 11.12 of [9]).

**Lemma 2.** Let

$$
\begin{array}{ccc}
X \xrightarrow{f} A & \Omega^k X \xrightarrow{\Omega^k f} \Omega^k A \\
\downarrow g & \downarrow \Omega^k g \\
B \xrightarrow{\nu} L & \Omega^k B \xrightarrow{\Omega^k \nu} L'
\end{array}
$$

where $g$ is $(n+k-1)$-connected and $\nu$ is $(m+n)$-connected.
denote the weak pushout squares (i.e. $L = C_{f,q}$ in the notation of [8]) and let $\Psi : L' \to \Omega^p L$ be the canonical map. Suppose $X$ is $(k+1)$-connected, $k \geq 1$. If $f$ is $p$-connected and $g$ is $q$-connected, $p \geq k+1, q \geq k+1$, then $\Psi$ is $(p+q-2k+1)$-connected.

**Corollary 1.** Let $h : U \to V$ be a $p$-connected map with $(q-1)$-connected $U$, $q-1 \geq k+1, p \geq k+1$. Then the canonical map $\Psi : C_{g,\text{rel}} \to \Omega^q C_{g,\text{rel}}$ is $(p+q-2k+1)$-connected.

**Lemma 3.** Let $f : A \to B$ be a map and let $\eta_A : A \to \Omega^k S^k A$ and $\eta_B : B \to \Omega^k S^k B$ denote the adjointness maps. If $f$ is $m$-connected, and if $A$ and $B$ are $(n-1)$-connected, $m \geq n \geq 1$, then the induced map $C_{\eta_A} \to C_{\eta_B}$ is $[n + \min(2n, m)]$-connected for $k \geq 2$ and $(n+m)$-connected for $k=1$.

**Proof.** Use Theorem 2.1 of Ganea [3], Corollary 1 and the relative Puppe sequences for $\Omega^{i'-i} A \to \Omega^{i'-i} A \to \Omega^{i'+1} A \to \Omega^{i'+1} A$ etc.

We say that $X$ is a homotopy $\Omega^k S^k$-space (or homotopy $\Omega^k S^k$-algebra) if there is a homotopy retraction $r : \Omega^k S^k X \to X$ of $\eta_k$ such that $r(\Omega^k S^k)$ is a homotopy $\Omega^k S^k$-space.

**Theorem 4.** Suppose $X$ is an $(n-1)$-connected homotopy $\Omega^k S^k$-space, $n \geq 2$. If $\pi_i(X) = 0$ for $i \geq 4n+1$, then $X$ has the homotopy type of a $k$-fold loop space.

**Proof.** Introduce the weak pushout squares

\[
\begin{array}{ccc}
S^k \Omega^k S^k X & \xrightarrow{\epsilon_k S^k} & S^k X \\
\downarrow r & & \downarrow i \\
S^k X & \xrightarrow{j} & L_1 \\
\end{array}
\quad \quad
\begin{array}{ccc}
\Omega^k S^k \Omega^k S^k X & \xrightarrow{\Omega^k S^k \epsilon} & \Omega^k S^k X \\
\downarrow j & & \downarrow j' \\
\Omega^k S^k X & \xrightarrow{\Delta} & L_2
\end{array}
\]

Then we have the maps $\Psi : L_1 \to \Omega^k L_1$, $\Phi : L_2 \to X$ such that $\Psi j = \Omega^k i, \phi j = r$. Since $r$ is $2n$-connected and $\epsilon_k S^k$ is $(2n+k)$-connected, we see from Lemma 2 that $\Psi$ is $(4n+1)$-connected, which implies $\theta \Psi \cong \Phi$ for a map $\Theta : \Omega^k L_1 \to X$. Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega^k S^k X & \xrightarrow{\eta'} & \Omega^k S^k \Omega^k S^k X \\
\downarrow r & & \downarrow j' \\
X & \xrightarrow{\eta'} & \Omega^k S^k X \\
\end{array}
\quad \quad
\begin{array}{ccc}
\Omega^k S^k \Omega^k S^k X & \xrightarrow{\Omega^k S^k \epsilon} & \Omega^k S^k X \\
\downarrow j & & \downarrow j' \\
\Omega^k S^k X & \xrightarrow{\Delta} & L_2 \\
\end{array}
\]

where $\eta'$ and $\eta$ denote $\eta_k$. Then $\theta \Psi j \eta' \cong \epsilon_1$ and $(\Omega^k \epsilon_k S^k) \eta' \cong \epsilon_1$. Let $\rho : C_{\eta} \to C_{\sigma}$ and $\sigma : C_{\eta} \to C_{\rho \epsilon \sigma}$ denote the induced maps. Since $\sigma$ is homeomorphic to $C_{\sigma}$ by virtue of the $3 \times 3$ lemma, and since $\rho$ is $3n$-connected by Lemma 3, we see that $\alpha$ is $3n$-connected. This shows that the induced map $C_{\eta} \to C_{\sigma}$ is $3n$-connected, since $C_{\rho \epsilon \sigma} \to C_{\eta}$ is a homotopy equivalence. It follows from the 5-lemma that $j \eta$ is $3n$-connected, hence $\Theta$ is $(3n+1)$-connected. Applying Lemma 1 to $\Theta$, we get a $(4n+1)$-connected map $X \to \Omega^k Y$, from which the theorem follows.

**Remark.** By duality we may prove that, if $X$ is an $(n-1)$-connected homotopy $S^k \Omega^k$-CW complex with $\dim X \leq 4n - 3k - 2, n \geq k+1 \geq 2$, then $X$ has the homotopy type of a $k$-fold suspension.
References