On the Characterization of the Linear Partial Differential Operators of Hyperbolic Type

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(Comm. by Kenjiro Shoda, M. J. A., July 12, 1973)

§1. Introduction. In this note we shall consider a linear partial differential operator $P(D)$ of degree $m$ with real constant coefficients in $n$ variables. By $\alpha$ we denote multi-indices, that is, $n$-tuples $(\alpha_1, \cdots, \alpha_n)$ of non-negative integers and by $|\alpha|$ their sum, that is $|\alpha| = \sum_{j=1}^{n} \alpha_j$. With $D_j = -\sqrt{-1} \partial / \partial x_j$, we set $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. Then the symbol $P(D)$ represents a differential operator $P(D) = \sum_{|\alpha| \leq m} a_{\alpha} D^\alpha$ and if $(\xi_1, \cdots, \xi_n) \in \mathbb{C}^n$, then $P(\xi)$ does the polynomial $P(\xi) = \sum_{|\alpha| \leq m} a_{\alpha} \xi^\alpha$. This gives a one-to-one correspondence between polynomials and differential operators with constant coefficients. We shall call the operator $P(D)$ irreducible if the polynomial $P(\xi)$ is irreducible.

The aim of this note is to characterize the linear partial differential operator $P(D)$ by the support of the solution $u(x) \in C^\infty(\mathbb{R}^n)$ of $P(D)u(x) = 0$. If $u(x)$ satisfies $P(D)u(x) = 0$, then $u(x)$ also satisfies $Q(D)P(D)u = 0$ for arbitrary differential operator $Q(D)$. So we shall consider only irreducible linear partial differential operators.

Cohoon [1] proved the following theorem:

Theorem A. There exists a nontrivial $u(x)$ in $C^\infty(\mathbb{R}^n)$ such that $P(D)u(x) = 0$ in $\mathbb{R}^n$ and such that the support of $u(x)$ is contained in $\{x \in \mathbb{R}^n; |x_k| \leq R, \text{ for } k = 1, 2, \cdots, n-1\}$ if and only if $P(D)$ is of the form $P(D) = aD^n + \sum_{|\alpha| < m} b_{\alpha} D^\alpha$ where $a (\neq 0)$ and $b_{\alpha} (|\alpha| < m)$ are real constants.

Then we ask when there exists a nontrivial $u(x)$ in $C^\infty(\mathbb{R}^n)$ such that $P(D)u(x) = 0$ in $\mathbb{R}^n$ and such that the support of $u(x)$ is contained in $\{x \in \mathbb{R}^n; |x_k| \leq R \text{ for } k = 1, \cdots, n-2 \text{ and } (r|x_n| + R)^2 - x_{n-1}^2 \geq 0 \}$ for $r \geq 0$. It is the purpose of this note to answer this question.

The author thanks Professor S. Koizumi for his helpful discussions to the material of this note.

§2. Definitions and theorem. By $P_m(D)$ we shall denote the principal part of $P(D)$. According to Hörmander [3] the operator $P(D)$ is called hyperbolic with respect to $N \in \mathbb{R}^n$, if $P_m(N) \neq 0$ and if there is a constant $\tau_0$ such that $P(\xi + i\tau N) \neq 0$, when $\tau < \tau_0$ and $\xi \in \mathbb{R}^n$. For the principal part $P_m(D)$ the definition of hyperbolicity is particularly simple by the following theorem.
Theorem. The principal part $P_m(D)$ of $P(D)$ is hyperbolic with respect to $N$ if and only if $P_m(N) \neq 0$ and the equation
\[ P_m(\xi + \tau N) = 0 \]
has only real roots when $\xi$ is real.

If $P(D)$ is hyperbolic with respect to $N$, we shall denote by $\Gamma(P, N)$ the set of all real $\theta$ such that polynomial $P_m(\theta + \tau N)$ has only negative root $\tau$. Then $\Gamma(P, N)$ is the component of $N$ in the open set $\{\theta; P_m(\theta) \neq 0\}$ and is a convex cone with vertex at 0. By $C^*$ we shall denote dual cone $\{x \in R^n; \langle x, \theta \rangle \geq 0, \theta \in C\}$ of cone $C$.

Let $e$ be the vector $(0, \ldots, 0, 1) \in R^n$. Let us introduce the domain $T_r = \{x \in R^n; |x_k| \leq R, k=1, \ldots, n-2$ and $(r|x_n| + R)^2 - x_n^2 \geq 0\}$, two cones $C_r = \{x \in R^n; x_n - (rx_{n-1})^2 > 0, x_n > 0\}, C_r' = \{x \in R^n; x_n - (rx_{n-1})^2 > 0, x_n < 0\}$ and the half space $H_N = \{x \in R^n; \langle x, N \rangle \geq 0\}$.

We shall prove the following theorem.

Theorem. Suppose $P(D)$ is an irreducible linear partial differential operator of degree $m$. Then there exists a nontrivial $u(x)$ in $C^\infty(R^n)$ such that (i) $P(D)u(x) = 0$ in $R^n$, (ii) the support of $u(x)$ is contained in $T_r$ if and only if $P(D)$ is of the form
\[ P(D) = a \sum_{i=1}^{m} (D_n + b_i D_{n-1}) + c(a)D_n, \quad |b_i| \leq r, \]
where $a(\neq 0), b_i(i=1, \ldots, m)$ and $c(a) (|a| < m)$ are real constants.

Theorem A is obtained by setting $r=0$ in this theorem. We show this theorem as a consequence of following two lemmas.

Lemma 1. There exists a nontrivial $u(x)$ in $C^\infty(R^n)$ which satisfies (i) and (ii) if and only if the cone $C_r$ is contained in $\Gamma(P_m, e)$.

Lemma 2. The cone $C_r$ is contained in $\Gamma(P_m, e)$ if and only if $P(D)$ is of the form (1).

§3. Proofs of Lemma 1 and Lemma 2. We first assume that $C_r \subset \Gamma(P_m, e)$. Let $\phi(x)$ be a $C^\infty$ function of Gevrey class $\delta(1 < \delta < m/m - 1)$ with the support in $\{x \in R^n; |x_k| \leq R, k=1, \ldots, n\}$. By the lemma 5.7.4 of Hörmander [3], there exists a function $U_k(\xi', x_n)$ which satisfies
\[ P(\xi, D_n)U_k(\xi', x_n) = 0, \quad \xi' = (\xi_1, \ldots, \xi_{n-1}) \]
\[ D_x^j U_k(\xi', 0) = 0, \quad 0 \leq j, k < m \quad \text{and} \quad j \neq k, \]
\[ D_x^k U_k(\xi', 0) = 1, \quad 0 \leq k < m, \]
and for some constant $K$
\[ |D^k_x U_k(\xi', x_n)| \leq K^{1+l}(|\xi'| + 1)^{l+m - k} \exp \left[K|x_n|(|\xi'| + 1)^{1-l/m}\right] \]
when $(\xi', x_n) \in R^n$ and $l=0, 1, 2, \ldots$.

Now let us consider
\[ v(\xi', x_n) = \sum_{k=0}^{m-1} (D_n^k \delta_n(\xi', 0))U_k(\xi', x_n) \]
where $\delta_n(\xi', x_n) = \int_{-\infty}^{x_n} e^{-i<\xi', x'>} \phi(x)dx'$. Using (3), (4), (5) and Paley-Wiener theorem, it follows that...
\[ D_k^m \psi(x', x_n) \leq \sum_{k=0}^{m-1} |D_k^k \phi_n(x', 0)| \cdot |D_n U_k(x', x_n)| \]
\[ \leq \sum_{k=0}^{m-1} K\rho K^{j+1}(1 + |x'|)^{j+m-k} \exp \left[ K|x_n|(|\xi'| + 1)^{1-1/m} B|\xi'|^{1-m} \right] \]
\[ \leq C\rho K^{j+1}(1 + |x'|)^{j+m} \exp \left[ (K|x_n| - B)|\xi'|^{1-1/m} \right] \]
for some constant \( C \) and \( B \geq R \). In particular this shows that \( \psi(x', x_n) \) is in \( L^1(R^{n-1}) \) for fixed \( x_n \). We can set \( u(x) = \mathcal{F}^{-1}[\psi(x', x_n)] \) where \( \mathcal{F}^{-1} \) is a partial inverse Fourier transform with respect to \( \xi_1, \ldots, \xi_{n-1} \).

Since \( B \) can be chosen arbitrary large, from (7) it follows that
\[ \|u(x', x_n)\|_{L^1} < \|1 + |\xi'|^{1} v(\xi', x_n)\|_{L^1} < \infty. \]

Since \( s \) can be chosen arbitrary large, from (8) and Sobolev’s lemma, we have
\[ u(x', x_n) \in C^s(R^{n-1}). \]

From this and (5), it follows that \( u(x) \in C^s(R^n) \).

Furthermore, from (2), (3), (4) and (6) we have
\[ P(D)u(x) = 0, \quad \text{in } R^n \]
and
\[ D_j u(x', 0) = D_j^j \phi(x', 0), \quad 0 \leq j < m. \]

Since \( \supp \phi(x', 0) \subset \{ x \in R^n ; |x_k| \leq R, k = 1, \ldots, n-1 \text{ and } x_n = 0 \} \), if we apply Corollary 5.3.2 of Hörmander [3], we can obtain,
\[ \supp U \cap H_\varepsilon \subset \{ x \in R^n ; |x_k| \leq R, k = 1, \ldots, n-1 \text{ and } x_n = 0 \} + \Gamma(P_m, -e)^\ast. \]

Similarly we have
\[ \supp U \cap H_{(-e)} \subset \{ x \in R^n ; |x_k| \leq R, k = 1, \ldots, n-1 \text{ and } x_n = 0 \} + \Gamma(P_m, e)^\ast. \]

Since \( \Gamma(P_m, e)^\ast \subset C_r^\ast \) and \( \Gamma(P_m, -e)^\ast \subset C_r^\ast \), we have \( \supp U \subset T_r \).

To prove the converse we consider the hyperplane \( \Sigma(N) = \{ x \in R^n ; \langle x, N \rangle = 0 \} \), where \( N \) is a vector in \( C_r \). It is obvious that \( \Sigma(N) \cap T_r \) is compact. Then we have \( N \in \Gamma(P_m, e) \). Because by the theorem of John [2], unless \( N \in \Gamma(P_m, e) \), \( u \) vanishes identically in \( \Sigma(N) \) and by translations, it follows that \( u \) vanishes identically in \( R^n \), which contradicts the assumption. This completes the proof of Lemma 1.

Proof of Lemma 2. We first assume that \( C_r \subset \Gamma(P_m, e) \). Let \( N \) be the vector such that \( N = (0, \ldots, N_{n-1}, N_n) \in C_r \). Let us consider the following equation with respect to \( \xi \).
\[ P_m(\xi_1, \xi_2, \ldots, \xi_{n-2}, \xi_{n-1}, N_{n-1}, N_n) = 0. \]

Suppose that for some \( (\xi_1, \ldots, \xi_{n-2}, 0, 0) \in R^n \) we could find nonzero complex number \( \zeta \) which satisfies (11). But Theorem 5.5.3 of Hörmander [3] tells us that \( \zeta \) must have been real.

Then we have
\[ (\xi_1, \xi_2, \ldots, \xi_{n-2}, \xi_{n-1}, N_{n-1}, N_n) \in C_r. \]

From this and the assumption we conclude that
\[ (\xi_1, \xi_2, \ldots, \xi_{n-2}, \xi_{n-1}, N_{n-1}, N_n) \in C_r. \]
is a hyperbolic direction of $P_m(D)$ and consequently that
\begin{equation}
P_m(\xi_1, \ldots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n) \neq 0.
\end{equation}
This contradicts that $\zeta$ is a root of equation of (11). Thus it is proved
that the equation
\begin{equation}
P_m(\xi_1, \ldots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n) = 0
\end{equation}
has only $\zeta = 0$ as a root. Furthermore
\begin{equation}
P_m(\xi_1, \ldots, \xi_{n-2}, \zeta N_{n-1}, \zeta N_n) = 
\sum_{|\alpha| + \beta + \gamma} a_{\alpha \beta \gamma} \xi^{\alpha} \eta^{\beta} \zeta^{\gamma} \zeta^{m}.
\end{equation}
We have
\begin{equation}
\sum_{|\alpha| + \beta + \gamma} a_{\alpha \beta \gamma} \xi^{\alpha} \zeta^{\gamma} \zeta^{m} = 0
\end{equation}
where $k = 0, \ldots, m-1, (0, \ldots, 0, N_{n-1}, N_n) \in C_r$.
Let us set $\eta = N_{n-1} N_n$ by $N_n \neq 0$. Since $C_r$ is a cone, we have
\begin{equation}
\sum_{|\alpha| + \beta + \gamma} a_{\alpha \beta \gamma} \eta^{\beta} \eta^{m} = 0
\end{equation}
for all $\eta''$ in $R^{n-2}$ and $\gamma$ in $(-r^{-1}, r^{-1})$. From this we conclude that
$a_{\alpha \beta \gamma} = 0$ for all $\alpha$ in $N^{n-2}$ with $|\alpha| = m - (\beta + \gamma)$ for all $(\beta, \gamma)$ with $0 \leq \beta + \gamma \leq m-1$ and $\beta \geq 0, \gamma \geq 0$. Thus $P_m(\xi) = Q(\xi_{n-1}, \xi_n)$ for some suitable
homogeneous polynomial of degree $m$ in two variables of $\xi_{n-1}$ and $\xi_n$. Then by the fundamental theorem of algebra, we can find the complex
numbers $a$ and $b_i (i = 1, \ldots, m)$ such that,
\begin{equation}
P_m(\xi) = a \sum_{i=1}^{m} (\xi_n + b_i \xi_{n-1}), \text{ where } a \neq 0.
\end{equation}
Since $e$ is a hyperbolic direction of $P_m(D)$, the $b_i (i = 1, \ldots, m)$ are real
constants. Let $c$ and $d$ be Max{$b_i$; $b_i \geq 0$}, Min{$b_i$; $b_i \geq 0$}, respectively. Then we have
\begin{equation}
\Gamma(P_m, e) = \{ x \in R^n ; x_n + cx_{n-1} > 0, x_n + dx_{n-1} > 0 \}.
\end{equation}
By the assumptions, it follows that $e \leq r$, $d \geq -r$. Thus, $|b_i| \leq r$, for
$i = 1, \ldots, m$.
Conversely, if $P(D)$ is of the form (1) then using (18), we conclude
that $C_r \subset \Gamma(P_m, e)$. The proof of Lemma 2 is complete.

References

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