9. On the Completions of Maps

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In this paper all spaces are assumed to be completely regular $T_2$. Let $f$ be a continuous map from a space $X$ into a space $Y$. As is well known, there exists its extension $\hat{\beta}(f) : \hat{\beta}(X) \to \hat{\beta}(Y)$, where $\hat{\beta}(S)$ denotes the Stone-Čech compactification of a space $S$. Furthermore, it is known that $\hat{\beta}(f)$ carries $\mu(X)$ into $\mu(Y)$ and $\nu(X)$ into $\nu(Y)$ ([14], [3]), where $\mu(X)$ is the topological completion of $X$ (that is, the completion of $X$ with respect to its finest uniformity $\mu$) and $\nu(X)$ is the realcompactification of $X$. We denote the restriction maps $\hat{\beta}(f) | \mu(X)$ and $\hat{\beta}(f) | \nu(X)$ by $\mu(f)$ and $\nu(f)$ respectively.

The purpose of this paper is to study the relations between $f$ and $\mu(f)$ (or $\nu(f)$).

We note first that $\mu(f) : \mu(X) \to \mu(Y)$ and $\nu(f) : \nu(X) \to \nu(Y)$ are not necessarily perfect even if $f : X \to Y$ is perfect. A continuous map $f$ from a space $X$ onto a space $Y$ is called a quasi-perfect (perfect) map if $f$ is a closed map such that $f^{-1}(y)$ is countably compact (resp. compact) for each $y \in Y$.

Example. Let $Y$ be a pseudo-compact space such that the preimage $X$ of $Y$ under a perfect map $f$ is not pseudo-compact ([4, Example 4.2]). Then both $\mu(f) : \mu(X) \to \mu(Y)$ and $\nu(f) : \nu(X) \to \nu(Y)$ are not perfect, since $\mu(X)$ and $\nu(X)$ are not compact, while $\mu(Y)$ and $\nu(Y)$ are compact (cf. [14], [3]).

In view of these results, it is significant to study under what conditions $\mu(f)$ (or $\nu(f)$) is perfect.

Theorem 1. If $f : X \to Y$ is an open quasi-perfect map, then $\mu(f) : \mu(X) \to \mu(Y)$ and $\nu(f) : \nu(X) \to \nu(Y)$ are open perfect.

To prove this theorem, we use the following lemmas.

Lemma 2 (Zenor [17]). Let $C(X)$ be the space of all the non-empty compact sets in a space $X$ with the finite topology. If $X$ is completely regular then so is $C(X)$.

The finite topology of $C(X)$ is defined as follows: For any finite number of open sets $\{U_1, \ldots, U_n\}$ of $X$, we set $\bigcup_{i=1}^n U_i = \{K \in C(X) | K \subset \bigcup_{i=1}^n U_i, K \cap U_i \neq \emptyset \}$ for $i = 1, \ldots, n$. As an open base of $C(X)$ we take all such sets. It is well known that if $X$ is completely regular then so is $C(X)$ (Michael [12]).
Lemma 3. If \( f : X \to Y \) is an open quasi-perfect map, then \( \varphi : Y \to C(\mu(X)) \) and \( \varphi^* : Y \to C(\nu(X)) \) are continuous, where \( \varphi(y) = \text{cl}_{\mu(X)} f^{-1}(y) \) and \( \varphi^*(y) = \text{cl}_{\nu(X)} f^{-1}(y) \) for each \( y \in Y \).

Hoshina [5] proved the continuity of \( \varphi : Y \to C(\mu(X)) \), and the continuity of \( \varphi^* : Y \to C(\nu(X)) \) is similarly proved.

Proof of Theorem 1. We note first that a surjective map \( g : X \to Y \) is perfect if and only if any filter base \( \{F_a\} \) in \( X \) such that \( \{g(F_a)\} \) has a cluster point in \( Y \) has a cluster point in \( X \). Now we prove the theorem for the case of \( \mu(f) \), since the case of \( \nu(f) \) is similarly proved. Let \( \mathcal{F} = \{F_a\} \) be a filter base in \( \mu(X) \) such that \( \{\mu(f)(F_a)\} \) has a cluster point \( v \) in \( \mu(Y) \). Let us put

\[
\mathcal{O} = \{G_r | v \in G_r, G_r : \text{open in } \mu(Y)\},
\]

\[
\mathcal{O}_Y = \{H_r | H_r = G_r \cap Y, G_r \in \mathcal{O}\}.
\]

Then \( \mathcal{O}_Y \) is a Cauchy filter base in \( Y \) with respect to \( \mu \), and it converges to \( v \) in \( \mu(Y) \). Since \( \varphi : Y \to C(\mu(X)) \) is continuous by Lemma 3, \( \{\varphi(H_r)\} \) is a Cauchy filter base in \( C(\mu(X)) \) with respect to the finest uniformity, and hence by Lemma 2 \( \{\varphi(H_r)\} \) converges to some \( K \in C(\mu(X)) \). Suppose that \( (\cap \text{cl}_{\mu(X)} F_a) \cap K = \emptyset \). Then for each point \( u \) of \( K \) there exists \( F_{a(u)} \) of \( \mathcal{F} \) such that \( u \in \mu(X) - \text{cl}_{\mu(X)} F_{a(u)} \). Therefore there exists a finite number of points \( \{u_1, \ldots, u_n\} \) of \( K \) such that

\[
K \subset \bigcup_{i=1}^{n} (\mu(X) - \text{cl}_{\mu(X)} F_{a(u_i)}),
\]

since \( K \) is compact. Let \( F_{\beta} \) be an element of \( \mathcal{F} \) such that \( F_{\beta} \subset F_{a(u_i)} \), \( i = 1, \ldots, n \). Then we have

\[
\bigcup_{i=1}^{n} (\mu(X) - \text{cl}_{\mu(X)} F_{a(u_i)}) \cap F_{\beta} = \emptyset.
\]

Let \( O \) be a regularly open set in \( \mu(X) \) such that

\[
K \subset O \subset \text{cl}_{\mu(X)} O \subset \bigcup_{i=1}^{n} (\mu(X) - \text{cl}_{\mu(X)} F_{a(u_i)}).
\]

Since \( \{f^{-1}(H_r)\} \) converges to \( K \) in \( C(\mu(X)) \), we have \( f^{-1}(H_r) \subset O \) for some \( H_r \in \mathcal{O}_Y \), and hence \( \mu(f)^{-1}(G_r) \subset O \). This shows that \( \mu(f)^{-1}(G_r) \cap F_{\beta} = \emptyset \), that is, \( G_r \cap \mu(f)(F_{\beta}) = \emptyset \), which is a contradiction. Therefore we have \( (\cap \text{cl}_{\mu(X)} F) \cap K = \emptyset \). Consequently \( \mathcal{F} \) has a cluster point in \( \mu(X) \). Moreover over the fact mentioned above it is easily seen that \( \mu(f) : \mu(X) \to \mu(Y) \) is surjective. Hence \( \mu(f) : \mu(X) \to \mu(Y) \) is perfect. Finally, by [10, Theorem 4.4], \( \beta(f) : \beta(X) \to \beta(Y) \) is an open map. Therefore it follows that \( \mu(f) \) is an open map. Thus we complete the proof.

Corollary 4. Let \( f : X \to Y \) be an open perfect map. Then the following statements are valid.

(a) \( Y \) is topologically complete if and only if \( X \) is topologically complete.

(b) \( Y \) is realcompact if and only if \( X \) is realcompact (Frolík [2]).

This corollary follows from Theorem 1 and the fact that the pre-
image of a topologically complete (realcompact) space under a perfect map is also topologically complete (resp. realcompact).

A continuous map \( f \) from a space \( X \) onto a space \( Y \) is called a WZ-map (Isiwata [10]) if \( \beta(f)^{-1}(y) = \text{cl}_{\beta(X)} f^{-1}(y) \) for each \( y \in Y \). Every closed map is a WZ-map. The following is a slight generalization of Theorem 1.

**Theorem 5.** If \( f : X \rightarrow Y \) is an open WZ-map such that \( f^{-1}(y) \) is relatively pseudo-compact for each \( y \in Y \), then \( \mu(f) : \mu(X) \rightarrow \mu(Y) \) and \( \nu(f) : \nu(X) \rightarrow \nu(Y) \) are open perfect.

**Proof.** Let \( X_0 = \beta(f)^{-1}(Y) \). Since \( \beta(f)^{-1}(y) = \text{cl}_{\beta(X)} f^{-1}(y) \) and \( \text{cl}_{\beta(X)} f^{-1}(y) \) is compact, we have \( X \subseteq X_0 \subseteq \mu(X) \subseteq \nu(X) \). Hence it follows that \( \mu(X_0) = \mu(X) \) and \( \nu(X_0) = \nu(X) \) ([14], [3]). On the other hand, \( \mu(f) : \mu(X_0) \rightarrow \mu(Y) \) and \( \nu(f) : \nu(X_0) \rightarrow \nu(Y) \) are open perfect by Theorem 1, since \( \beta(f)|X_0 : X_0 \rightarrow Y \) is an open perfect map. Thus the theorem holds.

**Corollary 6.** Let \( f : X \rightarrow Y \) be an open WZ-map such that \( f^{-1}(y) \) is relatively pseudo-compact for each \( y \in Y \). Then the following statements are valid.

(a) \( Y \) is pseudo-compact if and only if \( X \) is pseudo-compact.

(b) \( Y \) is pseudo-paracompact (pseudo-Lindelöf) if and only if \( X \) is pseudo-paracompact (resp. pseudo-Lindelöf).

Following Morita [14], a space \( X \) is said to be pseudo-paracompact (resp. Lindelöf) if \( \mu(X) \) is paracompact (resp. Lindelöf). In Corollary 6, (a) was proved by Isiwata [10] as a generalization of a theorem of Okuyama and Hanai [16], and the ‘only-if’ part of (b) was proved by Hoshina [5].

Concerning a (not necessarily open) quasi-perfect map, Morita [14] proved the following: If \( f \) is a quasi-perfect map from an \( M \)-space \( X \) onto an \( M \)-space \( Y \), then \( \mu(f) : \mu(X) \rightarrow \mu(Y) \) is a perfect map. As a generalization of this result, we can prove the following theorem.

**Theorem 7.** Let \( X \) and \( Y \) be the spaces each of which is the pre-image of a topologically complete space under a quasi-perfect map. If \( f : X \rightarrow Y \) is a quasi-perfect map, then \( \mu(f) : \mu(X) \rightarrow \mu(Y) \) is a perfect map.

To prove Theorem 7, we use the following lemmas.

**Lemma 8** (Ishii [9]). If \( f \) is a quasi-perfect map from a space \( X \) onto a topologically complete space \( Y \), then \( \mu(f) : \mu(X) \rightarrow Y \) is perfect.

**Lemma 9** (Kljusin [6]). Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be surjective. If \( h = g \circ f : X \rightarrow Z \) is perfect, then \( f \) and \( g \) are perfect.

**Proof of Theorem 7.** Let \( g : Y \rightarrow Z \) be a quasi-perfect map from \( Y \) onto a topologically complete space \( Z \). Then \( h = g \circ f : X \rightarrow Z \) is a quasi-perfect map, and hence \( \mu(h) : \mu(X) \rightarrow Z \) is a perfect map by Lemma 8. Let \( Y_0 = \mu(f)(\mu(X)) \). Since \( \mu(h) = \mu(g \circ f) = \mu(g) \circ \mu(f) \), \( \mu(g) \mid Y_0 : Y_0 \rightarrow Z \) is perfect by Lemma 9. Hence it follows that \( Y_0 \) is topologically...
complete, which implies that \( Y_0 = \mu(Y) \). Therefore \( \mu(f) : \mu(X) \to \mu(Y) \) is perfect by Lemma 9. Thus we complete the proof.

**Corollary 10.** Let \( X \) and \( Y \) be the spaces each of which is the preimage of a topologically complete space under a quasi-perfect map, and let \( f : X \to Y \) be a quasi-perfect map. Then \( Y \) is pseudo-paracompact (pseudo-Lindelöf) if and only if \( X \) is pseudo-paracompact (resp. pseudo-Lindelöf).

**Remark.** By Lemma 8, a space \( X \) is the preimage of a paracompact space under a quasi-perfect map if and only if \( X \) is a pseudo-paracompact space which is the preimage of a topologically complete space under a quasi-perfect map.

Applying Theorem 7, we can prove the following theorem.

**Theorem 11.** Let \( Y \) be an \( M^* \)-space ([7]). Then the following statements are equivalent.

(a) \( Y \) is the preimage of a topologically complete space under a quasi-perfect map.

(b) \( Y \) is an \( M \)-space.

**Proof.** Since (b) \( \implies \) (a) is obvious, we shall prove (a) \( \implies \) (b). Since \( Y \) is an \( M^* \)-space, there exists a perfect map \( f \) from an \( M \)-space \( X \) onto \( Y \) by Nagata's theorem [15]. Hence by Theorem 7 \( \mu(f) : \mu(X) \to \mu(Y) \) is a perfect map. Since \( \mu(X) \) is a paracompact \( M \)-space by Morita's theorem [14] and the image of a paracompact \( M \)-space under a perfect map is also a paracompact \( M \)-space (cf. Fillipov [1], Ishii [7], [8] and Morita [13]), \( \mu(Y) \) is a paracompact \( M \)-space. This implies that \( Y \) is an \( M' \)-space ([14]). Since each \( M^* \)-space is countably paracompact ([7]), \( Y \) is an \( M \)-space ([11]). Thus we complete the proof.

We note that Theorem 11 is also deduced directly from Lemma 8.

References


