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In this note we state a theorem on (micro-) analyticity of the elementary solutions of hyperbolic differential equations with (not necessarily constant) multiple characteristics. Our result is a generalization of those of Kawai [1], Hörmander [2] and Andersson [3] which deal with operators with simple characteristics. (See Atiyah-Bott-Gårding [4] for operators with constant coefficients.)

If an m-th order differential operator \( P(t, x, D_t, D_x) \) is hyperbolic with respect to the direction \((1, \cdots, 0)\), there exists a unique elementary solution of the Cauchy problem, that is, \( m \)-tuple of hyperfunctions \( E_j(t, x) (j=1, \cdots, m) \) such that

\[
P(t, x, D_t, D_x)E_j(t, x) = 0,
\]

\[
D_{t-i}^j E_j(0, x) = \delta_{ij} \delta(x) \quad (i, j = 1, \cdots, m).
\]

(See Kawai [5] and Bony-Schapira [6].) Our problem is to decide the singular spectrum of \( E_j(t, x) \).

Recently Kashiwara-Kawai [7] defined micro-hyperbolicity and constructed good elementary solutions for micro-hyperbolic operators. The essential key to our theorem is their deep analysis in micro-local sense. Remark that our lemma is valid for pseudo-differential operators.

Here we treat only the simplest case. More complete results and proofs will be published elsewhere.

First we set up a class of operators which can be easily handled. Let \( P(x, D_x) \) be a pseudo-differential operator defined in a neighborhood of \( x_0^* = (x_0, \xi_0) \in \mathbb{P}^*X \). Let \( a(P)(x, \xi) = p_1^1(x, \xi) \cdots p_r^1(x, \xi) \) be an irreducible decomposition at \( x_0^* \). We call \( P(x, D_x) \) reductive if each \( p_j(x, \xi) \) is simple characteristic, that is, \( d_j(x, \xi)p_j(x, \xi) \) is not parallel to \( \sum_1^r \xi_1 dx_1 \). In this case we can define \( r \)-bicharacteristic strips through \( x_0^* \). A hyperbolic differential operator is called reductive if it is reductive at each point on its real characteristic variety.

Examples.

\[
D_t^2 - \ell_1 (D_x^2 + D_y^2)
\]

\[
(D_t^2 - a(t, x, y)D_x^2 - b(t, x, y)D_y^2)(D_t^2 - c(t, x, y)D_x^2 - d(t, x, y)D_y^2)
\]

where \( a, b, c \) and \( d \) is positive for real \((t, x, y)\).
Now let us define characteristic conoid of a reductive hyperbolic differential operator. Let \( P(t, x, D_t, D_x) \) be a reductive hyperbolic differential operator with respect to the direction \((1,\ldots,0)\) defined in a neighborhood of the origin. Let \( \mathcal{V}_R = \{(t, x, \tau, \xi) \in \sqrt{-1}S^*M/\sigma(P)(t, x, \tau, \xi) = 0\} \) and \( D = \{(t, x, \tau, \xi) \in \mathcal{V}_R/ \text{the number of bicharacteristic strips through } (t, x, \tau, \xi) \geq 2\} \). We assume the following: at each point \((t_0, x_0, \tau_0, \xi_0) \in D\), the number of bicharacteristic strips is two and if \( \sigma(P)(t, x, \tau, \xi) = p_1(t, x, \tau, \xi)p_2(t, x, \tau, \xi) \) is an irreducible decomposition, \( \{p_1, p_2\}(t_0, x_0, \tau_0, \xi_0) \neq 0 \) where \( \{\, , \} \) denotes Poisson bracket. Let us pursue a bicharacteristic strip through \((0, 0, \tau_0, \xi_0) \in \pi^{-1}(0)\cap \mathcal{V}_R \) where \( \pi: \sqrt{-1}S^*M \rightarrow M \). It will fall across \( D \). Then two bicharacteristic strips come forth from there and they may again fall across \( D \) and so on. We call the union of these bicharacteristic strips the characteristic conoid.

**Theorem.** The elementary solution \( E_3(t, x) \) is micro-analytic except the characteristic conoid.

The essential part of the proof of this theorem is the following lemma.

**Lemma.** Let \( P(t, x, D_t, D_x) \) be a pseudo-differential operator defined in a neighborhood of \((0, \ldots, 0, \sqrt{-1}(0, \ldots, 1)\infty)\) such that \( \sigma(P) = t^n\tau^n \). If a microfunction \( u \) satisfies

a) \( P(t, x, D_t, D_x)u = 0 \),

b) \( u = 0 \) on \( \{(t, 0, \ldots, 0, \sqrt{-1}(0, \ldots, 1)\infty/ t < 0\} \cup \{(0, 0, \ldots, 0, \sqrt{-1}(t, 0, \ldots, 1)\infty/t < 0) \) or \( b' \) \( u = 0 \) on \( \{(t, 0, \ldots, 0, \sqrt{-1}(0, \ldots, 1)\infty/t < 0\} \cup \{(0, \ldots, 0, \sqrt{-1}(t, 0, \ldots, 1)\infty/t > 0) \), then \( u = 0 \) at \((0, \ldots, 0, \sqrt{-1}(0, \ldots, 1)\infty)\).

This is an easy corollary of the existence of a good elementary solution of Kashiwara-Kawai.

**References**


