2. Remarks on a Totally Real Submanifold

By Seiichi YAMAGUCHI and Toshihiko IKAWA
Department of Mathematics, Science University of Tokyo
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§ 1. Introduction. K. Yano and S. Ishihara [8] and J. Erbacher [3] have determined the submanifold $M$ of non-negative sectional curvature in the Euclidean space or in the sphere with constant mean curvature, such that $M$ has a constant scalar curvature and a flat normal connection.

Recently, C. S. Houh [4], S. T. Yau [9], and B. Y. Chen and K. Ogiue [2] have investigated totally real submanifolds in a Kähler manifold with constant holomorphic sectional curvature $c$.

On the other hand, the authors [5]-[7] studied $C$-totally real submanifolds in a Sasakian manifold with constant $\phi$-holomorphic sectional curvature. In particular, we have dealt with $C$-totally real submanifolds with flat normal connection in [6].

The purpose of this paper is to obtain the following:

Theorem. Let $M^n$ be a totally real submanifold in a Kähler manifold $\mathbb{M}^{n+p}$. A necessary and sufficient condition in order that the normal connection is flat is that the submanifold $M^n$ is flat.

§ 2. Preliminaries. Let $M^n$ be a submanifold immersed in a Riemannian manifold $\mathbb{M}^{n+p}$. Let $\langle , \rangle$ be the metric tensor field on $\mathbb{M}^{n+p}$ as well as the metric tensor induced on $M^n$. We denote by $\nabla$ the covariant differentiation in $\mathbb{M}^{n+p}$ and $\nabla$ the covariant differentiation in $M^n$ determined by the induced metric on $M^n$. Let $\mathfrak{K}(\mathcal{M})$ (resp. $\mathfrak{K}(\mathcal{M})$) be the Lie algebra of vector fields on $\mathcal{M}$ (resp. $\mathcal{M}$) and $\mathfrak{K}^\perp(\mathcal{M})$ the set of all vector fields normal to $\mathcal{M}$.

The Gauss-Weingarten formulas are given by

\begin{align}
\tag{2.1}
& \nabla_X Y = \nabla_X Y + B(X, Y), \\
\tag{2.2}
& \nabla_X N = -A^N(X) + D_X N, \quad X, Y \in \mathfrak{K}(\mathcal{M}), \quad N \in \mathfrak{K}^\perp(\mathcal{M}),
\end{align}

where $\langle B(X, Y), N \rangle = \langle A^N(X), Y \rangle$ and $D_X N$ is the covariant derivative of the normal connection. $A$ and $B$ are called the second fundamental form of $M$.

The curvature tensors associated with $\nabla, \nabla, D$ are defined by the followings respectively:

\begin{align}
& R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \\
& R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \\
& R^\perp(X, Y) = [D_X, D_Y] - D_{[X, Y]}.
\end{align}

If the curvature tensor $R^\perp$ of the normal connection $D$ vanishes, then
§ 3. Proof of Theorem. Let $\bar{M}^{2n}$ be a Kähler manifold with Kähler structure $J$. A submanifold $M^n$ in $\bar{M}^{2n}$ is called totally real submanifold if each tangent space of $M^n$ is mapped into the normal space by the Kähler structure $J$.

Now, we shall prove Theorem stated in § 1. Let $E_1, \ldots, E_n$ be orthonormal basis of $\mathfrak{T}(M)$, then by the definition of the totally real submanifold, $\mathfrak{T}^\perp(M)$ is spanned by $JE_1, \ldots, JE_n$. Therefore, if $N \in \mathfrak{T}^\perp(M)$, then $JN \in \mathfrak{T}(M)$. Since $JY \in \mathfrak{T}^\perp(M)$, it follows that

$$F_X(JY) = -A^{\gamma Y}(X) + D_X(JY),$$

by virtue of (2.2). On the other hand, we can obtain

$$F_X(JY) = J(F_X Y) = J(F_X Y) + JB(X, Y).$$

Comparing with the normal part of (3.1) and (3.2), we have

$$D_X(JY) = J(F_X Y).$$

Operating $D_Z$ to (3.3) and making use of (3.3), we can get

$$D_Z D_X(JY) = D_Z(J(F_X Y)) = JV_X F_X Y.$$

Interchanging the vectors $X$ and $Z$ in this equation and taking account of (2.3) and (3.3), it holds that

$$R^\perp(Z, X) JY = JR(Z, X) Y.$$

This completes the proof.

References