49. Automorphic Forms and Algebraic Extensions of Number Fields*1

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§ 0. The purpose of this paper is to present a result on an arithmetical relation between Hilbert cusp forms over a totally real algebraic number field, which is a cyclic extension of the rational number field \( \mathbb{Q} \) with a prime degree \( l \), and cusp forms of one variable. The details of this result will appear in [7].

Let \( F \) be a totally real algebraic number field, and \( \mathcal{O} \) be its maximal order. For an even positive integer \( \kappa \), let \( S_\kappa(F) \) denote the space of Hilbert cusp forms of weight \( \kappa \) with respect to the subgroup \( \Gamma = \text{GL}_2(\mathcal{O})^+ \) consisting of all elements with totally positive determinants in \( \text{GL}_2(\mathcal{O}) \). For a place (archimedean or non-archimedean) \( v \) of \( F \), let \( F_v \) be the completion of \( F \) at \( v \). For a non-archimedean place \( v \) (= \( p \)), let \( \mathcal{O}_v \) be the ring of \( p \)-adic integers of \( F_v \). Let \( F_A \) be the adele ring of \( F \), and consider the adele group \( \text{GL}_2(F_A) \). Let \( \mathcal{U}_F \) be the open subgroup \( \mathcal{U} = \text{GL}_2(\mathcal{O}_v) \times \text{GL}_2(F_v) \) of \( \text{GL}_2(F_A) \). Then we can consider the Hecke ring \( R(\mathcal{U}_F, \text{GL}_2(F_A)) \) and its action \( \Xi \) on \( S_\kappa(F) \) as in G. Shimura [8].

For the ordinary modular group \( \text{SL}_2(\mathbb{Z}) = \text{GL}_2(\mathbb{Z})^+ \), we also consider its adelization \( \mathcal{U}_\mathbb{Q} = \prod_p \text{GL}_2(\mathbb{Z}_p) \times \text{GL}_2(\mathbb{R}) \) and the Hecke ring \( R(\mathcal{U}_\mathbb{Q}, \text{GL}_2(\mathbb{Q}_A)) \). The latter is acting on the space \( S_\kappa(\text{SL}_2(\mathbb{Z})) \) of cusp forms of weight \( \kappa \) with respect to \( \text{SL}_2(\mathbb{Z}) \).

§ 1. The space \( S_\kappa(F) \). Suppose \( F \) is a cyclic extension of \( \mathbb{Q} \) of degree \( l \). We fix an embedding of \( F \) into the real number field \( \mathbb{R} \) and a generator \( a \) of the Galois group \( \text{Gal}(F/\mathbb{Q}) \) of the extension \( F/\mathbb{Q} \), then all the distinct embeddings of \( F \) into \( \mathbb{R} \) are given by \( \sigma^i, 0 \leq i \leq l-1 \). We consider the group \( \text{GL}_2(F) \) as a subgroup of \( \text{GL}_2(\mathbb{R}) \) by \( g \mapsto (g, \sigma^i g, \ldots, \sigma^{l-1} g) \) for \( g \in \text{GL}_2(F) \). For this fixed generator \( \sigma \), we define an operator \( T_\sigma \) on \( S_\kappa(F) \) by the permutation of variables, namely \( T_\sigma f(z_1, \ldots, z_l) = f(z_{\sigma^i}, \ldots, z_l, z_{\sigma^0}) \) for \( f \in S_\kappa(F) \). Using this \( T_\sigma \), we define a new subspace \( S_\kappa(F) \) of \( S_\kappa(F) \), to be called “the space of symmetric Hilbert cusp forms”, as follows:

\[
S_\kappa(F) = \{ f \in S_\kappa(F) | \Xi(e)T_\sigma f = T_\sigma e f \quad \text{for any} \quad e \in R(\mathcal{U}_F, \text{GL}_2(F_A)) \}.
\]

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Obviously $S_\ell(I')$ is stable under the action of $R(\mathfrak{l}_F, GL_2(\mathbb{A}_F))$, and we get a new representation $\tilde{\mathcal{A}}_S$ of the Hecke ring $R(\mathfrak{l}_F, GL_2(\mathbb{A}_F))$ on the space $S_\ell(I')$.

Now we assume

0) The weight $\kappa \geq 4$.
1) The degree $l=[F: \mathbb{Q}]$ is a prime.
2) The class number of $F$ is one.
3) The maximal order $\mathfrak{o}$ has a unit of any signature distribution.
4) $F$ is tamely ramified over $\mathbb{Q}$.

As a consequence of 2) and 4), the conductor of $F/\mathbb{Q}$ is a prime $q$.

Our result claims that the representation $\tilde{\mathcal{A}}_S$ of $R(\mathfrak{l}_F, GL_2(\mathbb{A}_F))$ on $S_\ell(I')$ can be obtained from those on the spaces of cusp forms $S_\ell(SL_2(\mathbb{Z}))$ and $S_\ell(I_0(q), \chi)$ for various characters $\chi$ of $(\mathbb{Z}/q\mathbb{Z})^\times$ of order $l$, where

$$\Gamma_\ell(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod q \right\},$$

and $S_\ell(I_\ell(q), \chi)$ is the space of cusp forms $g$ which satisfy $g(\gamma z) = \chi(d)(cz+d)^\ell g(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\ell(q)$.

To give a meaningful description for the above, we shall define a "natural" homomorphism $\lambda : R(\mathfrak{l}_F, GL_2(\mathbb{A}_F)) \to R(\mathfrak{l}_\mathbb{Q}, GL_2(\mathbb{Q}_F))$ in the next section § 2. On the other hand, $R(\mathfrak{l}_\mathbb{Q}, GL_2(\mathbb{A}_F))$ is acting not only on $S_\ell(SL_2(\mathbb{Z}))$ but also on $S_\ell(I_\ell(q), \chi)$. For a prime $p$, let $T(p)$ and $T(p, p)$ be the elements of $R(\mathfrak{l}_\mathbb{Q}, GL_2(\mathbb{Q}_F))$ given in § 2. Then for $p \neq q$, $T(p)$ and $T(p, p)$ act on $S_\ell(I_\ell(q), \chi)$ in the usual manner ([9]). For $q$, let $\Gamma_\ell(q)(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \Gamma_\ell(q) = \sum_{\chi} \alpha_\chi \Gamma_\ell(q)$ be a disjoint union, and put for $g \in S_\ell(I_\ell(q), \chi)$

$$g \left[ \Gamma_\ell(q)(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}) \Gamma_\ell(q) \right] = \sum_{\chi} \chi(d) \left( \begin{pmatrix} c & z \\ -c & c \end{pmatrix} \right) g(\alpha_\chi^{-1}z)$$

where $\alpha_\chi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. And we define the action of $T(q)$ and $T(q, q)$ on $S_\ell(I_\ell(q), \chi)$ by

$$T(q)g = g \left[ \Gamma_\ell(q)(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}) \Gamma_\ell(q) \right] + g \left[ \Gamma_\ell(q)(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}) \Gamma_\ell(q) \right]^*$$

$$T(q, q)g = g.$$

Here $\left[ \Gamma_\ell(q)(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}) \Gamma_\ell(q) \right]^*$ denotes the adjoint operator of $\left[ \Gamma_\ell(q)(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}) \Gamma_\ell(q) \right]$ with respect to the Petersson inner product. These actions of $T(p)$ and $T(p, p)$ can be extended to that of $R(\mathfrak{l}_\mathbb{Q}, GL_2(\mathbb{Q}_F))$, and we obtain the representations $\tilde{\mathcal{A}}_\ell$ and $\tilde{\mathcal{A}}_x$ of $R(\mathfrak{l}_\mathbb{Q}, GL_2(\mathbb{Q}_F))$ on $S_\ell(SL_2(\mathbb{Z}))$ and $S_\ell(I_\ell(q), \chi)$, respectively. Thus $S_\ell(SL_2(\mathbb{Z}))$ (resp. $S_\ell(I_\ell(q), \chi)$) can be viewed as a $R(\mathfrak{l}_F, GL_2(\mathbb{A}_F))$-module by the action $\mathcal{A}_\ell \lambda$ (resp. $\mathcal{A}_x \lambda$). In these notations, we can prove
Theorem. There exists a subspace $S$ of $\bigoplus_i S_i(\Gamma_0(q), \chi)$ such that
$$S_i(\Gamma) \simeq S_i(SL_2(\mathbb{Z})) \oplus S$$
(and $\bigoplus_i S_i(\Gamma_0(q), \chi) \simeq S \oplus S$)
as $R(\mathbb{I}_F, GL_2(F_A))$-modules, where in $\bigoplus_i \chi$ runs through all characters
of order 1 of $(\mathbb{Z}/q\mathbb{Z})^\times$.

This theorem can be derived by standard arguments from the following equality between the traces of the operators.

Theorem. $\text{tr} \mathcal{X}_S(e) = \text{tr} \mathcal{X}_i(\lambda(e)) + \frac{1}{2} \sum_i \text{tr} \mathcal{X}_i(\lambda(e))$

for any $e \in R(\mathbb{I}_F, GL_2(F_A))$.

§ 2. The homomorphism $\lambda: R(\mathbb{I}_F, GL_2(F_A)) \rightarrow R(\mathbb{I}_Q, GL_2(Q_A))$. Let $a$ (resp. $n$) be an integral ideal of $F$ (resp. a positive integer), and $T(a)$ (resp. $T(n)$) be the sum of all integral elements in $R(\mathbb{I}_F, GL_2(F_A))$ (resp. $R(\mathbb{I}_Q, GL_2(Q_A))$) of norm $a$ (resp. $n$). For a prime ideal $p$ of $F$ (resp. a prime $p$), let $T(p, p)$ (resp. $T(p, p)$) denote the double coset $U_F a U_F$ (resp. $U_Q a U_Q$), where the $p$-component (resp. $p$-component) of $a$ is $\left( \begin{array}{cc} \pi & 0 \\ 0 & \pi \end{array} \right)$ (resp. $\left( \begin{array}{cc} p & 0 \\ 0 & p \end{array} \right)$) with a prime element $\pi$ of $\mathcal{O}$, and the other component is the identity. We define elements $U(p^m)$ (resp. $U(p^m)$) of $R(\mathbb{I}_F, GL_2(F_A))$ (resp. $R(\mathbb{I}_Q, GL_2(Q_A))$) for a prime ideal $p$ of $F$ (resp. a prime $p$) and a non-negative integer $m$ by

$U(a) = 2T(a)$

(resp. $U(1) = 2T(1)$)

$U(p^m) = \begin{cases} T(p^m) - NpT(p, p)T(p^{m-2}), & m = 1 \\ T(p^m) - NpT(p, p)T(p^{m-2}), & m \geq 2 \end{cases}$

(resp. $U(p^m) = \begin{cases} T(p^m) - NpT(p, p)T(p^{m-2}), & m = 1 \\ T(p^m) - NpT(p, p)T(p^{m-2}), & m \geq 2 \end{cases}$)

where $Np$ is the cardinality of $\mathcal{O}/p$. Then the correspondence $U(p^m) \rightarrow U(Np^m)$ can be extended to a homomorphism $\lambda$ from $R(\mathbb{I}_F, GL_2(F_A))$ to $R(\mathbb{I}_Q, GL_2(Q_A))$.

§ 3. Applications. Our result is related to the recent works of the following authors.

(I) In their joint work [2], K. Doi and H. Naganuma studied a relation between cusp forms with respect to $SL_2(Z)$ and Hilbert cusp forms over real quadratic fields. More precisely, let $\varphi(s) = \sum a_n n^{-s}$, $a_1 = 1$, be the Dirichlet series associated with a cusp form of weight $k$ with respect to $SL_2(Z)$ which is a common eigen-function for all Hecke operators, and let $\chi$ be the real character corresponding to a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ in the sense of class field theory. If we put $\varphi(s, \chi) = \sum \chi(n) a_n n^{-s}$, then $\varphi(s) \varphi(s, \chi)$ can be expressed in the following form with suitable coefficients $C_\chi$ which are defined for every integral
ideal $\alpha$ in $F$;

$$\varphi(s)\varphi(s, \chi) = \sum_{\alpha} C_{\alpha} N\alpha^{-s}.$$ 

For a Grössen-character $\xi$ of $F$, we set

$$D(s, \varphi, \chi, \xi) = \sum_{\alpha} \xi(\alpha) C_{\alpha} N\alpha^{-s}.$$ 

In [2], K. Doi and H. Naganuma tried to prove a functional equation of $D(s, \varphi, \chi, \xi)$ and proved it for the case where the conductor of $\xi$ is one, and showed that if the maximal order of $F$ is an Euclidean domain, the Dirichlet series $\varphi(s)\varphi(s, \chi)$ is actually associated with a Hilbert cusp form over $F$ and the function

$$h(z_1, z_2) = \sum_{\psi/\sqrt{q} \neq 0} \psi \sum_{\alpha \in \mathcal{O}_\mathcal{F}} \exp \left( 2\pi i \left( \frac{\psi \mu}{\sqrt{q}} z_1 + \frac{\psi \mu}{\sqrt{q}} z_2 \right) \right)$$

on the product $\mathcal{H} \times \mathcal{H}$ of the complex upper half planes is a Hilbert cusp form over $F$. Moreover in [6], H. Naganuma showed that a similar result holds also for cusp forms of “Neben” type (in Hecke’s sense) with a prime level. Now from our present result for $l=2$, it can be proved that $\varphi(s)\varphi(s, \chi)$ is the Dirichlet series associated with a Hilbert cusp form over a real quadratic field $F$, and that Doi-Naganuma’s construction is “injective” under the condition of this paper.

(II) In [5], H. Jacquet studied the similar theme as Doi-Naganuma’s, in a more general (adic and representation-theoretic) point of view, hence this result should have a close connection to ours.

(III) F. Hirzebruch [3] [4] and R. Busam [1] gave a dimension formula for the subspace $S_\alpha(I')$ of $S_\alpha(I)$ consisting of elements $f$ such that $T_\alpha f = (-1)^{\nu_2} f$. Since there is an obvious relation

$$\dim S_\alpha(I') = \frac{1}{2} \left( \dim S_\alpha(I) + (-1)^{\nu_2} \dim S_\alpha(I') \right),$$

our result can be viewed as a generalization of their formula.

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References


