81. A Note on Isolated Singularity. II

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0. Introduction. This is a brief résumé of the second half of the study whose first part has already been announced [3]. The main purpose is to investigate the structure of an isolated singularity when it admits a $C^*$-action, especially, to obtain some formula concerning the characters of the representations of $C^*$ over various cohomology groups associated with the singularity.

1. Basic concepts. A $C^*$-action over an isolated singularity $(X, x)$ is a family $T(c), c \in C^*$ of analytic homeomorphisms of $X$ onto itself satisfying that $T(c)x = x$, $T(cc') = T(c)T(c')(c, c' \in C^*)$, and that $T : X \times C \ni (x, c) \rightarrow T(z)c \in X$ is analytic. Throughout this note we assume that the constants are the only invariant elements of $\Omega^p_{X,x}$ under this action. Let $\xi$ be the generating vector field of this action. The interior multiplication $i(\xi)$ is an anti-derivation of $\Omega^p_X$ regarded as the sheaf of graded algebra. It is well known that the Poincaré complex $\Omega^p_X$ is acyclic in this case. However we have some more

**Lemma 1.** Under the above condition the sequences
\[
\cdots \rightarrow \mathcal{H}^0_x(\Omega^p_X) \rightarrow \mathcal{H}^0_x(\Omega^{p+1}_X) \rightarrow \cdots \\
\cdots \rightarrow \Omega^p_X \rightarrow i(\xi) \Omega^{p-1}_X \rightarrow \cdots \rightarrow \Omega^p_{X,x} \ni f \rightarrow \Omega^p_{x}(\xi)f \rightarrow 0
\]
are exact, where $\iota_x$ denotes the inclusion $x \rightarrow X$ and $\alpha$ the average map

\[
\Omega^p_{X,x} \ni f \rightarrow \int_0^1 T(e^{2\pi i}\theta)^nf d\theta \in (\iota_x)_* \Omega^p_x.
\]

If we set $\Omega^p_x = i(\xi)\Omega^{p+1}_X$, then we have the short exact sequences
\[
0 \rightarrow \Omega^p_x \rightarrow \Omega^p_{X} \rightarrow \Omega^p_{X,x} \rightarrow 0.
\]
From these we obtain the following Gysin type sequences

\[
0 \rightarrow \mathcal{H}^0_x(\Omega^p_X) \rightarrow \mathcal{H}^0_x(\Omega^p_X) \rightarrow \mathcal{H}^0_x(\Omega^{p+1}_X) \rightarrow \cdots \\
\cdots \rightarrow \mathcal{H}^0_x(\Omega^p_X) \rightarrow \mathcal{H}^0_x(\Omega^p_X) \rightarrow \mathcal{H}^0_x(\Omega^{p+1}_X) \rightarrow \cdots
\]

Using these, we can prove

**Theorem 1.** Let the notation and the assumption be as above. Assume that $(X, x)$ satisfies the condition (L). Then $\mathcal{H}^0_x(\Omega^p_X) = 0$ for $(p, q)$ such that $p + q \neq \dim X$, $q \neq p + 1$, $q < \dim X$, and there are natural isomorphisms $\mathcal{H}^0_x(\Omega^p_X) \simeq \mathcal{H}^0_x(\Omega^{p+1}_X)$ for $(p, q)$ such that $p + q = \dim X, 0 < q < \dim X$.

**Remark.** If $\dim X$ is even, the proof requires some technique from Kähler geometry, though we can avoid the use of this in case
dim $X$ is odd.

2. Formula for characteristic function. In the rest of this note we assume that $(X, x)$ satisfies the condition (L). Let $0 \leq q < n = \dim X$ and denote by $\chi^X(t)$ the character of the representation of $C^*$ on $\mathcal{H}_2(\Omega_{X}^{n-q})$ (or on $\mathcal{H}_2(\Omega_{X}^{n-q+1})$ in view of Theorem 1 if $q > 0$); that is,

$$\chi^X(t) = \text{Tr} (T(t) | \mathcal{H}_2(\Omega_{X}^{n-q}))$$

where the notation in the largest parenthesis on the right denotes the endomorphism of $\mathcal{H}_2(\Omega_{X}^{n-q})$ induced by the action $T(t)$. These are rational functions in $t$. In view of the Serre type duality we have $\chi^X(t) = \chi^{X-q+1}(t^{-1})$ for $2 \leq q < n - 1$, so it is convenient to set $\chi^{X}(t) = \chi^{X}(t^{-1})$, $\chi^{X-n+1}(t) = \chi^{X}(t^{-1})$; further we define the characteristic function of the $C^*$-action by

$$\chi(s, t) = \sum_{q=0}^{n} \chi^X(t) s^{q}.$$  

Now let $f$ be an analytic function on $X$ such that $df \neq 0$ for $z \in X \setminus x$, $T(c)^* f = cdf$ ($d > 0$). Then $f(x) = 0$ and $(Y, y) = (f^{-1}(0), x)$ is a new isolated singularity over which the action $T(c)$, $c \in C^*$ induces a $C^*$-action. We consider $\chi^X(t)$, $\chi^Y(t)$, $\cdots$, $\chi^Y(t)$, $\chi(s, t)$ to be defined similarly. Then, using some argument in proving the result of [3], we obtain

**Theorem 2.** Let the assumption and the notation be as above. Then

$$s(\chi(s, t) - s^n \chi(s^{-1})) - t^d(\chi(s, t) - \chi(t)) = (t^d - 1)(\chi(s, t) - \chi(t) - s^n \chi(s^{-1})).$$

**Remark.** The characters $\chi^X(t)$, $\chi^Y(t)$ are in a sense computable; for example, according to Lemma 1, $\chi^X(t)$ is equal to an alternating sum of the characters on the spaces $\Omega_{X}^p$, $p > n$; to determine these spaces from the defining equation of $(X, x)$ is easier comparing with the determination of the cohomology groups $\mathcal{H}_2(\Omega_{X}^p)$.

3. Application. Let $H^r(X \setminus x, C)$, $H^r(Y \setminus y, C)$ be the fixed part of $\mathcal{H}_2(\Omega_{X}^q)$, $\mathcal{H}_2(\Omega_{X}^r)$ with respect to the action $T$; further define $H^{n-1} \mathcal{H}_2(X \setminus x, C)$, $H^{n-2} \mathcal{H}_2(X \setminus y, C)$ respectively. Then, as an application of Theorem 2, we have

**Corollary 1.** There are canonical direct sum decompositions

$$H^r(X \setminus x, C) = \sum_{p+q=r} H^p_q$$

and natural exact sequences

$$0 \longrightarrow H^{n-1}(X \setminus x, C) \longrightarrow H^n(X \setminus X, C) \longrightarrow H^n(Y \setminus y, C) \longrightarrow 0$$

$$0 \longrightarrow H^n(X \setminus x, C) \longrightarrow H^n(X \setminus y, C) \longrightarrow H^n(Y \setminus y, C) \longrightarrow 0.$$

**Remark.** These direct sum decompositions also arise from the mixed Hodge structures of $X \setminus x$, $Y \setminus y$ in the sense of [1]. Thus, in this case, the characteristic function explains some aspect of the mixed
Hodge structure. Note also that this corollary shows the degeneracy of the spectral sequence $E^{p,q}_{2}(Y, y) = H^{p}(R^{q}g^{*}Ω^{\gamma}_{Y})(\gamma : Y \to Y)$, and of $E^{p,q}_{2}(X, x)$ defined similarly.

Now we shall apply Theorem 1 and Theorem 2 to the study of algebraic manifolds. Let $V$ be an $n$-dimensional submanifold in $P_{n+r}(C)$ and $E$ the line bundle over $V$ induced by the hyperplane sections. Since $E^{-1}$ is negative, we can consider the quotient space $C(V) = E^{-1}/V$ by shrinking the zero section $V$ into a point $p$, and we thus have isolated singularity $(C(V), p)$. Now we assume that $V$ is the intersection of $r$ non-singular hypersurfaces of $P_{n+r}(C)$ which are situated in a general position. Then $C(V)$ is a complete intersection, so it satisfies condition (L). (See [3].) Note that, on $C(V)$, there is the natural $C^{*}$-action induced by the multiplication of $C$ in the line bundle $E^{-1}$. We consider the functions $\chi_{C(V)}(t)$ ($0 \leq q \leq n+2$), $\chi_{C(V)}(s, t)$ to be defined with respect to this action. Now let $h^{p,q}(E^{k})$ be the dimension of $H^{p}(V, Ω^{q}(E^{k}))$ and let the polynomials $R^{i}(z_{1}, z_{2}, \ldots, z_{i})$ $i=1, 2, \ldots$ be defined inductively by $R^{i}(z) = (z-1)^{n+r+1}, R^{i+1}(z_{1}, z_{2}, \ldots, z_{i+1}) = (z_{1}R_{1}(z_{2}, z_{3}, \ldots, z_{i+1}) - z_{2}R_{1}(z_{1}, z_{3}, \ldots, z_{i+1}))(z_{1}-z_{2})$. Then we have

**Corollary 2.** The assumptions being as above, $H^{q}(V, Ω^{p}(E^{k})) = 0$ if $p+q \neq n$, $0 < q < n$, $k \neq 0$; further, if $a_{1}, \ldots, a_{r}$ are the degrees of the hypersurfaces defining $V$, then the following congruence holds

$$\chi_{C(V)}(s, t) = \chi_{C(V)}(t) + \frac{s}{t^{a_{1}} - s} R^{s}_{a_{1}} - s \prod_{j=1}^{r} \frac{t^{a_{j}+1} - 1}{t - 1} \left( \frac{t^{a_{j}+1} - 1}{t - 1} \right)$$

where the last term should be regarded as a power series in $s$ whose coefficients are rational functions in $t$. Moreover $\chi_{C(V)}(t) - \sum_{k \leq n} h^{n,q}(E^{k})t^{k}$ is a polynomial divisible by $t^{n}$.

This corollary, combined with Theorems 22.1.1–22.1.2 of Hirzebruch [2], determines all of the dimensions of $H^{q}(V, Ω^{p}(E^{k}))$.

The details will appear elsewhere.

**References**
