88. The Baire Category Theorem in Ranked Spaces

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In this note, we study the Baire category theorem for a ranked space of indicator \( \omega_0 \) (\( \omega_0 \) is the first nonfinite ordinal). Throughout this note, the term “ranked space” will mean a ranked space of indicator \( \omega_0 \). Terminologies and notations concerning ranked spaces will be the same as in [5], in particular, \( N \) will denote the set \( \{0, 1, 2, \ldots\} \), \( V(p) \), \( W(p) \), \ldots preneighborhoods of \( p \), and \( V(p, n), W(p, n), \ldots \) those of rank \( n \) of \( p \).

1. The Baire category theorem. For a ranked space, we define the notion of nowhere dense as follows.

Definition 1. Let \( (E, CV) \) be a ranked space. A subset \( A \) of \( E \) is said to be nowhere dense in \( E \) if, for every \( V(p) \in CV \), there exists a \( V(q) \in CV \) such that \( V(q) \subset V(p) \) and \( V(q) \cap A = \emptyset \).

Moreover, as in [2] we define:

Definition 2. For a ranked space \( (E, CV) \), a subset \( A \) of \( E \) is said to be of first category if it is a countable union of nowhere dense sets. All other subsets of \( E \) are said to be of second category. A subset \( A \) of \( E \) is said to be dense in \( E \) if, for every \( V(p) \in CV \), we have \( V(p) \cap A 
eq \emptyset \). The ranked space \( (E, CV) \) is called a Baire space if, for every subset \( A \) of \( E \) which is of first category, the complement \( E - A \) is dense in \( E \).

As is easily seen, if \( (E, CV) \) is a ranked space for which we can topologise \( E \) in such a way that the family of all sets belonging to \( CV \) is a base of neighborhoods, then the notion of Baire category in \( (E, CV) \) coincides with that in the topological space \( E \) topologised in this way.

We first prove the following theorem.

Theorem 1. Every complete ranked space is a Baire space.

Already, for a ranked space whose indicator is an arbitrary inaccessible limit ordinal, the same theorem has been proved by K. Kunugi [2], [4] under the assumption that the family \( CV \) of preneighborhoods in the ranked space satisfies the following conditions (B) and (C).

(B) For every \( V_1(p), V_2(p) \in CV \), there exists a \( V_3(p) \in CV \) such that \( V_3(p) \subset V_1(p) \cap V_2(p) \).

(C) For every \( V(p) \in CV \), if \( q \in V(p) \), then there exists a \( V(q) \in CV \) such that \( V(q) \subset V(p) \).

Theorem 1 asserts that if we define nowhere dense as in Definition
1 and if the indicator of the ranked space is $\omega_0$, then Kunugi’s result holds without the assumptions of (B) and (C).

Proof of Theorem 1. Let $(E, CV)$ be a complete ranked space. Let $A = \bigcup_{i=1}^{\infty} H_i$, where each $H_i$ is nowhere dense in $E$, and let $V(p) \in CV$. We will show that $V(p) \cap (E - A) \neq \emptyset$. We first put $G_i = E - H_i$ for all $i$. Then, since $H_i$ is nowhere dense in $E$, there exists a $V(q_0) \in CV$ such that $V(q_0) \subset V(p)$ and $V(q_0) \subset G_0$. Also, by the axiom (a) of ranked space, there exists a $V(q_0, n_0) \in CV$ such that $V(q_0, n_0) \subset V(q_0)$. Thus, for $V(p)$, we may take a $V(q_0)$ such that $V(q_0, n_0) \subset V(p) \cap G_0$. Moreover, by the axiom (a), we may take a $V(q_1, n_1) \in CV$ such that $V(q_1, n_1) \subset V(q_0, n_0)$, $q_1 = q_0$, and $n_1 > n_0$. Suppose that $V(q_j, n_j)$ ($j = 0, 1, 2, \ldots, 2i - 1$) have been chosen such that $V(q_j, n_j) \subset V(q_{j-1}, n_{j-1})$, $q_j = q_{j-1}$, and $n_j > n_{j-1}$, and $V(q_j, n_j) \subset V(p) \cap G_j$ for $0 \leq j \leq i - 1$. Then, since $H_i$ is nowhere dense in $E$, we may take, as in the case of $i = 0$, a $V(q_{2i}, n_{2i}) \in CV$ such that $V(q_{2i}, n_{2i}) \subset V(q_{2i-1}, n_{2i-1}) \cap G_i$ and $n_{2i} > n_{2i-1}$, and a $V(q_{2i+1}, n_{2i+1}) \subset V(q_{2i}, n_{2i})$, $q_{2i+1} = q_{2i}$, and $n_{2i+1} > n_{2i}$. We thus obtain a fundamental sequence $\{V(q_i, n_i)\}$ such that $\cap V(q_i, n_i) \subset V(p) \cap (\cap G_i)$. Hence, $V(p) \cap (E - A) \neq \emptyset$ follows from the completeness of $(E, CV)$.

Example 1 (due to K. Kunugi [4]). Let $R^2$ be the 2-dimensional Euclidean space and let $p \in R^2$, $p = (x_0, y_0)$. For each $n \in N$ and for each real number $l$ such that $2 < l < +\infty$, we denote by $V(p; n, l)$ the set \[\{(x, y); 0 \leq (x-x_0)(y-y_0) < 1/n + 1, 0 \leq x-x_0 < l, 0 \leq y-y_0 < l\},\] by $CV_n(p)$ the family of all $V(p; n, l)$ such that $2 \leq l < +\infty$, and by $CV(p)$ the family $\cup \{CV_n(p); n \in N\}$. Then, $(R^2, CV, CV_0)$, where $CV = \cup \{CV(p); p \in R^2\}$ and $CV_0 = \cup \{CV_n(p); p \in R^2\}$, is a complete ranked space which does not satisfy (C*) (see 2 below) weaker than (C).

2. Characterizations of Baire spaces. We give some definitions which are needed for other characterizations of Baire spaces.

Definition 3. Let $(E, CV)$ be a ranked space, and let $A$ be a subset of $E$. Then, $A$ is called open if, for every $p \in A$, there exists a $V(p) \in CV$ such that $V(p) \subset A$. A is called closed if $E - A$ is open. The set $\cap \{O; O$ is open, $O \subset A\}$ is called the interior of $A$ and denoted by $A^i$. The set $\cap \{F; F$ is closed, $A \subset F \subset E\}$ is called the closure of $A$ and denoted by $A^c$.

Moreover, for $(E, CV)$, we consider the following condition.

For every $V(p) \in CV$, there exists a $W(p) \in CV$ such that $W(p) \subset V(p)$ and such that, for every $q \in W(p)$, there exists a $V(q) \in CV$ such that $V(q) \subset V(p)$.

Then, we have

Proposition 1. If, for a ranked space $(E, CV)$, $CV$ satisfies (C*), then a subset $A$ of $E$ is nowhere dense in $E$ if and only if $A^c$ is nowhere dense.
Proposition 2. For a ranked space \((E, \mathcal{C})\), let us consider the following.

(a) \((E, \mathcal{C})\) is a Baire space.

(b) Every countable intersection of open dense sets in \(E\) is dense in \(E\).

(c) For every countable family \(F_n (n=1, 2, \ldots)\) of closed sets satisfying \(E = \bigcup F_n\), \(\bigcap (F_n)^c\) is dense in \(E\).

Then, we have: (1) If \(\mathcal{C}\) satisfies (B) and (C*), then (a) implies (b) and (c); (2) If \(\mathcal{C}\) satisfies (C*), then each of (b) and (c) implies (a).

The proofs of these propositions are similar to those of the corresponding results in topological spaces.

3. Complete ranked spaces and \(\alpha\)-favorable topological spaces (due to G. Choquet [1]). As a technique for deciding when a given topological space is Baire, G. Choquet [1] has introduced the notion of \(\alpha\)-favorable, stemming from game theory, and proved that every \(\alpha\)-favorable topological space is a Baire space. The following proposition shows the connection between the notion of completeness in ranked spaces and the notion of \(\alpha\)-favorable.

Proposition 3. Let \(E\) be a topological space for which we can define a complete ranked space \((E, \mathcal{C})\) such that (1): \(\mathcal{C}\) is a family consisting of neighborhoods in \(E\) which forms a base for the topology of \(E\), furthermore \(\mathcal{C}\) has the property (2): there exists a \(k \in \mathbb{N}\) such that if, for \(V(p, n), V(q, m) \in \mathcal{C}, V(p, n) \supseteq V(q, m)\) and \(V(q, m) \neq \{q\}\), then \(n \leq m + k\). Then, \(E\) is \(\alpha\)-favorable.

Proof. We define a map \(f\) of \(\mathcal{C}\) into \(\mathcal{C}^*\) in such a way that: if \(V(p) \in \mathcal{C}\), then \(f(V(p))\) is a \(V(p, n) \in \mathcal{C}\) for which there exists a \(V(p, m) \in \mathcal{C}\) such that \(V(p, n) \supseteq V(p, m) \supseteq V(p)\) and \(m + k < n\). The existence of such a \(V(p, n)\) follows from the axiom (a) of ranked space. We will prove that if \(\{V(p_{2i})\}; i=0,1,2,\ldots\) is a sequence of neighborhoods defined inductively so that \(V(p_{2i}) \supseteq V(p_{2i+1}) = f(V(p_{2i})) \supseteq V(p_{2i+2}) = f(V(p_{2i+1})) \supseteq \ldots\), then \(\bigcap V(p_{2i}) \neq \emptyset\). We put \(f(V(p_{2i})) = V(p_{2i}, n)\). Then, we may obtain a sequence \(\{V(p_{2i}, m_{2i})\}; i=0,1,2,\ldots\) of neighborhoods such that (1°): \(m_{2i} + k < n_{2i}\) for all \(i\), and such that \(V(p_{2i}, n_{2i}) \supseteq V(p_{2i}, m_{2i}) \supseteq V(p_{2i})\) for all \(i\), and therefore (2°): \(V(p_{0}, m_{0}) \supseteq V(p_{0}, n_{0}) \supseteq \cdots \supseteq V(p_{2i}, m_{2i}) \supseteq V(p_{2i}, n_{2i}) \supseteq \cdots\). In (2°), if \(V(p_{2i}, m_{2i}) \neq \{p_{2i}\}\) for all \(i\), then by (2) and (1°), we have \(m_{0} + k < n_{0} \leq m_{2} + k < n_{2} \leq \cdots\), and so a subsequence of (2°) is fundamental. Hence, \(\bigcap V(p_{2i}, n_{2i}) \neq \emptyset\). If, in (2°), there exists an \(i_0\) such that \(V(p_{2i_0}, m_{2i_0}) = \{p_{2i_0}\}\), then \(\bigcap V(p_{2i}, n_{2i}) = \{p_{2i}\}\). Thus, \(\bigcap V(p_{2i}) \neq \emptyset\) follows.

\(^{a}\) We remark that [1], 7.13 holds under the assumption that \(\mathcal{C}^*\) in [1], 7.11 is a base of neighborhoods.
The following examples are topological spaces satisfying the assumptions of Proposition 3.

Example 2. Complete metric spaces.

Let $E$ be a complete metric space with a distance function $d$ and let $p \in E$. We denote the set $\{q \in E; d(p,q) < 1/2^n\}$ by $S(p,n)$. If $p$ is an isolated point of $E$, we put $V(p) = \{p\}$ and define $C^V_n(p) = \{V(p)\}$ for all $n \in N$. If not, there exists a subsequence of $N: n_0(p) < n_1(p) < \cdots < n_k(p) < \cdots$ such that $n_k(p) = 0$ and such that, for every $k$, $S(p, n_{k+1}(p))$ is a proper subset of $S(p, n_k(p))$ and $S(p, n_{k+1}(p)) = S(p, n)$ for all $n_k(p) \leq n < n_{k+1}(p)$. Using $\{n_k(p)\}$, we define $C^V_n(p)$ as follows. For $n \in N$, if $n = n_k(p)$ for some $k \in N$, $C^V_n(p) = \{S(p, n)\}$, that is, $S(p, n)$ is the only preneighborhood of rank $n$ of $p$; otherwise, $C^V_n(p) = \emptyset$. Then, $(E, C^V, C^V_n)$ is a desired ranked space if we put $C^V = \bigcup \{C^V(p); p \in E\}$, where $C^V(p) = \bigcup \{C^V_n(p); n \in N\}$, and put $C^V_n = \bigcup \{C^V_n(p); p \in E\}$ (cf. [3], Theorem 1).

Example 3. Cartesian products of the real lines, endowed with the product topology.

In this case, the ranked space obtained by putting $V(x_1, x_2, \ldots, x_n; m) = \{f(x); |f(x_i)| < 1/2^m\}$ in [3], Example, is a desired ranked space.

References