168. The Isometry Groups of Compact Manifolds with Non-positive Curvature^{*)}

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Let M be an n -dimensional compact connected Riemannian manifold with negative Ricci curvature. Then a classical theorem of Bochner says that there exist no non-trivial Killing vector fields on M . And hence the order of the isometry group $I(M)$ of M is finite. Relating to this theorem, T. Frankel obtained the following:

Let M be a compact Riemannian manifold with non-positive sectional curvature $K_s \leq 0$ and with negative Ricci curvature. If $f: M \rightarrow M$ is an isometry which is continuously homotopic to the identity map, then f is the identity, see [1]. This result was extended by H.B. Lawson and S. T. Yau in a more general situation. That is

Theorem ([5; Theorem 4, p. 225]). *Let M be a compact Riemannian manifold with non-positive sectional curvature and Ricci curvature negative at some point of M . If $f: M \rightarrow M$ is an isometry continuously homotopic to the identity, then f is the identity.*

As a corollary of this theorem, we easily have

Lemma 1. *Let M be a manifold as in the theorem of Lawson and Yau. If $f: M \rightarrow M$ is an isometry such that $d(p, f(p)) < d(p, C(p))$ for all point $p \in M$, then f is the identity.*

Here d is the distance function of M induced from the Riemannian metric and $C(p)$ the cut locus of p in M .

Now, for such manifolds as in the theorem of Bochner or Lawson and Yau, it is natural to ask whether we can estimate the order of the isometry group $I(M)$ by using the geometrical terms of M , for example, the diameter, the injectivity radius, the sectional curvature and so on. To this problem, H. C. Im Hof gave an estimation of order of $I(M)$ for a manifold with the sectional curvature K_s satisfying $-b^2 \leq K_s \leq -a^2 < 0$, $0 < a \leq b$. In his argument, the assumption that M is of negative curvature is essential.

In this note, we will give an estimation of the order of $I(M)$ for manifolds as in the theorem of Lawson and Yau in a different way from the one in H. C. Im Hof's theorem. The author thanks Prof. T. Otsuki for his kind advices.

Let M be a compact Riemannian manifold. For a point $p \in M$,

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$T_p(M)$ denotes the tangent space of M at p and $\exp_p: T_p(M) \rightarrow M$ the exponential mapping. Every geodesic is parametrized with respect to the arc-length. Put $\delta := \min \{d(p, C(p)) : p \in M\}$. δ is called the injectivity radius of M . For a positive number r and a point $p \in M$, $B_r(p)$ denotes the open metric ball in M with radius r centered at p . By the compactness of M , we can choose a finite number of points $p_1, \dots, p_k \in M$ such that

$$(*) \quad \bigcup_{i=1}^k B_{\delta/4}(p_i) = M.$$

Of course k depend on δ and other geometrical structure of M . In this situation, we have

Proposition 1. *Let M be a compact Riemannian manifold with non-positive sectional curvature and Ricci curvature negative at some point of M . Then*

$$\text{the order of } I(M) \leq k^k.$$

Proof. We define a mapping $\Phi: I(M) \rightarrow \{\text{all the } k\text{-sequences } (i_1, i_2, \dots, i_k), 1 \leq i_1, i_2, \dots, i_k \leq k\}$ as follows: for $f \in I(M)$

$$\Phi(f) = (j_1, j_2, \dots, j_k)$$

where for each $i=1, \dots, k$, the number j_i is the smallest of j such that $f(p_i) \in B_{\delta/4}(p_j)$. Φ is well defined, because by means of (*). We will show that Φ is injective. Assume $\Phi(f) = \Phi(g) = (j_1, j_2, \dots, j_k)$ for $f, g \in I(M)$. Then we have

$$d(f(p_i), p_{j_i}) < \frac{\delta}{4} \quad \text{and} \quad d(g(p_i), p_{j_i}) < \frac{\delta}{4}, \quad i=1, \dots, k.$$

Hence, putting $h := f^{-1} \circ g$, for each $i=1, \dots, k$, we have

$$\begin{aligned} d(p_i, h(p_i)) &= d(p_i, f^{-1} \circ g(p_i)) = d(f(p_i), g(p_i)) \\ &\leq d(f(p_i), p_{j_i}) + d(p_{j_i}, g(p_i)) < \frac{\delta}{2}. \end{aligned}$$

Now for each point $p \in M$, we can find a number i_0 such that $p \in B_{\delta/4}(p_{i_0})$ by (*). So, from the above fact, we have

$$\begin{aligned} d(p, h(p)) &\leq d(p, p_{i_0}) + d(p_{i_0}, h(p_{i_0})) + d(h(p_{i_0}), h(p)) \\ &< \frac{\delta}{4} + \frac{\delta}{2} + d(p_{i_0}, p) < \frac{\delta}{4} + \frac{\delta}{2} + \frac{\delta}{4} = \delta. \end{aligned}$$

Thus for each point $p \in M$, $d(p, h(p)) < \delta \leq d(p, C(p))$. Then by Lemma 1, h must be identity. So our assertion is proved. q.e.d.

Next, we will calculate the number k for compact manifolds with the sectional curvature satisfying $-b^2 \leq K_s \leq 0$ for some $b > 0$.

Let H be the n -dimensional hyperbolic space of constant curvature $-b^2$ represented on the Euclidean disk $\{x \in R^n : \|x\|^2 < 1/b^2\}$ with the

Riemannian metric $ds^2 = \frac{4}{(1 - b^2 \|x\|^2)^2} \sum_{i=1}^n dx_i^2$. For a positive number t ,

$[t]$ denotes the integer part of t . Let $r, s, r > s$ be any positive numbers.

Lemma 2. For the point $\bar{p}=(0, \dots, 0) \in H$, there exists a finite number of points $\bar{p}_1, \dots, \bar{p}_l \in B_r(\bar{p})$ such that

$$\bigcup_{i=1}^l B_s(\bar{p}_i) \supset \overline{B_r(\bar{p})}.$$

The number l can be taken as

$$l \leq \left(\left\lceil \frac{r}{s} \right\rceil + 1 \right) \left(\left\lceil \frac{2R}{\alpha} \right\rceil + 1 \right)^n + 1$$

where R and α are determined with β and a as follows:

$$(**) \quad \begin{cases} \cosh b\beta = \cosh bs / \cosh \frac{bs}{2}, & a = \tanh \frac{br}{2} / b, \\ R = 2a / (1 - a^2 b^2), & \alpha = \left\{ \frac{2R^2}{n} \left(1 - \cos \frac{\beta}{R} \right) \right\}^{1/2}. \end{cases}$$

Proof. Let $\gamma: [0, \infty) \rightarrow H$ be a geodesic starting from $\bar{p}$. Consider a covering of $\gamma([0, r])$:

$$\bigcup_{j=0}^{\lceil r/s \rceil + 1} B_s(\gamma(j \cdot s)) \supset \gamma([0, r]).$$

Let β be the number satisfying (**). Then, by using the hyperbolic trigonometry, we can see that β -tubular neighborhood

$$U_\beta(\gamma([0, r])) := \{ \bar{q} \in H : d(\bar{q}, \gamma([0, r])) < \beta \}$$

of γ is contained in $\bigcup_j B_s(\gamma(j \cdot s))$. And we have

$$U_\beta(\gamma([0, r])) \cap \partial B_r(\bar{p}) \supset B_{\beta'}^{\gamma(r)}(\gamma(r)) \supset B_{\beta'}^{\gamma(r)}(\gamma(r))$$

for some $\beta' > \beta$, where $B_{\beta'}^{\gamma(r)}(\gamma(r))$ denotes the open metric ball in $\partial B_r(\bar{p})$ with radius β' centered at $\gamma(r)$. Note that $\partial B_r(\bar{p})$ is isometric to the standard $(n-1)$ -dimensional sphere $S^{n-1}(R)$. For any point $\bar{q} \in B_{\beta'}^{\gamma(r)}(\gamma(r))$ and the geodesic $\eta: [0, r] \rightarrow H$ connecting from $\gamma(0)$ to $\bar{q}$, we easily see that $\eta([0, r]) \subset U_\beta(\gamma([0, r]))$. Thus, if we can find a finite number of points $\bar{p}_0, \dots, \bar{p}_m \in S^{n-1}(R)$ such that

$$\bigcup_{i=1}^m B_{\beta'}^{\gamma(r)}(\bar{p}_i) \supset S^{n-1}(R),$$

then, from the above fact, we have

$$\overline{B_r(\bar{p})} \subset \bigcup_{i=1}^m U_\beta(\gamma_i([0, r])) \subset \bigcup_{i=1}^m \bigcup_{j=0}^{\lceil r/s \rceil + 1} B_s(\gamma_i(j \cdot s)),$$

where $\gamma_i: [0, r] \rightarrow H$ is the geodesic from $\bar{p}$ to $\bar{p}_i, i=1, \dots, m$.

Now, for any point $p=(p_1, \dots, p_n) \in R^n$ and a positive number $\rho > 0$, we set

$$I(p, \rho) := \{ x=(x_1, \dots, x_n) \in R^n : 0 < |x_i - p_i| < \rho/2, 1 \leq i \leq n \}.$$

Then we can easily see

$$\begin{aligned} & \sup \{ d(q, q') : q, q' \in I(p, \rho) \cap S^{n-1}(R) \} \\ & \leq R \arccos \frac{2R^2 - n\rho^2}{2R^2} \quad \text{if } n\rho^2 \leq 2R^2. \end{aligned}$$

where d is the distance function on $S^{n-1}(R)$. We choose $\alpha > 0$ such that

$$R \arccos \frac{2R^2 - n\alpha^2}{2R^2} = \beta$$

$$\text{i.e. } \alpha = \left\{ \frac{2R^2}{n} \left(1 - \cos \frac{\beta}{R} \right) \right\}^{1/2}.$$

Thus, if we can find a finite number of points $q_1, \dots, q_m \in R^n$ such that

$$\bigcup_{i=1}^m \overline{I(q_i, \alpha)} \supset S^{n-1}(R) \quad \text{and} \quad \overline{I(q_i, \alpha)} \cap S^{n-1}(R) \neq \emptyset, i=1, 2, \dots, m$$

and taking a point $\bar{p}_i \in \overline{I(q_i, \alpha)} \cap S^{n-1}(R)$ for each i , then, from the above fact, we have

$$S^{n-1}(R) = \bigcup_{i=1}^m \overline{I(q_i, \alpha)} \cap S^{n-1}(R) \subset \bigcup_{i=1}^m \overline{B_{\beta}^{n-1}(\bar{p}_i)} \subset \bigcup_{i=1}^m B_{\beta'}^{n-1}(\bar{p}_i).$$

Hence, by means of an Euclidean geometric consideration, we can choose an m with the above mentioned property such that

$$m \leq \begin{cases} \left(\left\lfloor \frac{2R}{\alpha} \right\rfloor + 1 \right)^n & \text{if } \left\lfloor \frac{2R}{\alpha} \right\rfloor \neq \frac{2R}{\alpha} \\ \left\lfloor \frac{2R}{\alpha} \right\rfloor^n & \text{if } \left\lfloor \frac{2R}{\alpha} \right\rfloor = \frac{2R}{\alpha}. \end{cases} \quad \text{q.e.d.}$$

Remark. The estimation of the number m in the proof of Lemma 2 is very rough. We counted the number of points $q_i \in R^n$ such that $\overline{I(q_i, \alpha)} \cap S^{n-1}(R) = \emptyset$. When α is small with respect to R , then, for example, we can choose an upper bound of m as

$$m \leq \begin{cases} \left\{ 2 \left(\left\lfloor \frac{R}{\alpha} \right\rfloor + 1 \right) \right\}^n - \left\{ 2 \left\lfloor \frac{R}{\sqrt{n}\alpha} \right\rfloor \right\}^n & \text{if } \left\lfloor \frac{R}{\alpha} \right\rfloor \neq \frac{R}{\alpha} \\ \left(2 \left\lfloor \frac{R}{\alpha} \right\rfloor \right)^n - \left(2 \left\lfloor \frac{R}{\sqrt{n}\alpha} \right\rfloor \right)^n & \text{if } \left\lfloor \frac{R}{\alpha} \right\rfloor = \frac{R}{\alpha}. \end{cases}$$

And it will be possible to get a more sharp estimation.

Using this Lemma, we have

Proposition 2. Let M be an n -dimensional compact Riemannian manifold with the diameter d and the injectivity radius δ , whose sectional curvature K_s satisfies $-b^2 \leq K_s \leq 0$ for some $b > 0$. Then there exists a finite number of points $p_1, \dots, p_k \in M$ such that

$$\bigcup_{i=1}^k B_{\delta/4}(p_i) = M \quad \text{with} \quad k \leq \left(\left\lfloor \frac{4d}{\delta} \right\rfloor + 1 \right) \left(\left\lfloor \frac{2R}{\alpha} \right\rfloor + 1 \right)^n + 1$$

where R and α are determined as in Lemma 2 by replacing r and s with d and $\delta/4$ respectively.

Proof. Let $\pi: \tilde{M} \rightarrow M$ be the universal covering of M and π its projection. Then as is well known, $\tilde{M}$ is diffeomorphic to an n -dimensional Euclidean space and each geodesic segment of M is the shortest connection between its end points. Fix a point $p \in M$ and $\tilde{p} \in \{\pi^{-1}(p)\}$. Clearly $B_d(\tilde{p})$ contains a fundamental domain D of this covering i.e. $\pi: D \rightarrow M$ is one-to-one and $\pi: \tilde{D} \rightarrow M$ is onto. Since δ is the injectivity radius of M , for each point $\tilde{q} \in \tilde{M}$, $\pi: B_{\delta/4}(\tilde{q}) \rightarrow B_{\delta/4}(\pi(\tilde{q}))$ is onto and isometric. So, if we can find a finite number of points $\tilde{p}_1, \dots, \tilde{p}_k \in B_d(\tilde{p})$ satisfying $\bigcup_{i=1}^k B_{\delta/4}(\tilde{p}_i) \supset \overline{B_d(\tilde{p})}$, then we have $\bigcup_{i=1}^k B_{\delta/4}(\pi(\tilde{p}_i)) = M$. Let H and

$\bar{p} \in H$ be the space and the point as in Lemma 2. Let $l: T_{\bar{p}}(H) \rightarrow T_{\bar{p}}(\tilde{M})$ be an isometry. And consider a mapping

$$\varphi := \exp_{\bar{p}} \cdot l \cdot \exp_{\bar{p}}^{-1}: H \rightarrow \tilde{M}.$$

For any point $\bar{q} \in B_a(\bar{p})$, let $\gamma: [0, \delta/4] \rightarrow H$ be a geodesic starting from $\bar{q}$. Then, by Rauch's Comparison Theorem (see [2: Korollar p. 179]),

$$L(\gamma) \geq L(\varphi(\gamma))$$

where L denotes the length of a curve. So $\varphi(B_{\delta/4}(\bar{q})) \subset B_{\delta/4}(\varphi(\bar{q}))$ for any $\bar{q} \in B_a(\bar{p})$. Thus, if we find a finite number of points $\bar{p}_1, \dots, \bar{p}_k \in H$ such that $\bigcup_{i=1}^k B_{\delta/4}(\bar{p}_i) \supset \bar{B}_a(\bar{p})$, then, from the above facts, we have

$$\bar{B}_a(\bar{p}) = \overline{\varphi(B_a(\bar{p}))} \subset \bigcup_{i=1}^k \overline{\varphi(B_{\delta/4}(\bar{p}_i))} \subset \bigcup_{i=1}^k B_{\delta/4}(\varphi(\bar{p}_i)).$$

And the existence of such points $\bar{p}_i$ is already shown in Lemma 2.

q.e.d.

Summerising the above,

Theorem. *Let M be an n -dimensional compact connected Riemannian manifold with non-positive sectional curvature K_s ; $-b^2 \leq K_s \leq 0$ for some $b > 0$ and with Ricci curvature negative at some point. Let d and δ be the diameter and the injectivity radius of M respectively. Then*

the order of $I(M) \leq k^k$,

where k is given by $k = \left(\left\lceil \frac{4d}{\delta} \right\rceil + 1\right) \left(\left\lceil \frac{2R}{\alpha} \right\rceil + 1\right)^n + 1$ and R, α are determined with β, α as follows:

$$\cosh b\beta = \cosh \frac{b\delta}{4} / \cosh \frac{b\delta}{8}, \quad a = \tanh \frac{bd}{2} / b,$$

$$R = 2a / (1 - a^2 b^2), \quad \alpha = \left\{ \frac{2R^2}{n} \left(1 - \cos \frac{\beta}{R} \right) \right\}^{1/2}.$$

References

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