16. An Interpolation of Operators in the Martingale $H_p$-spaces

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1. Introduction. In this note we show that the Marcinkiewicz interpolation theorem of operators can be extended in the martingale setting.

2. Definition. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^\infty)$ a probability space furnished with a non-decreasing sequence of $\sigma$-algebras of measurable subsets $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots \subset \mathcal{F} = \bigvee_{n=1}^\infty \mathcal{F}_n$.

We define the set of random variables $H_p = H_p(\Omega, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^\infty) = \left\{ f \in L^p(\Omega); \| f \|_p = \left[ \int_\Omega (f^*)^p dP \right]^{1/p} < \infty \right\}$, where $f^*(w) = \sup_{1 \leq n < \infty} |f_n(w)|$ and $p \geq 1$.

Note that $H_1 \subseteq L^1$, and that $H_p = L^p$ for $1 < p < \infty$. In fact, there exists a constant $A_p$ such that $\| f \|_p \leq A_p \| f \|_1$. Furthermore, as is well-known, the norm $\| f \|_p$ is equivalent to $\| (\sum_{n=1}^\infty |f_n|^p)^{1/p} \|_p$, where $f_n = f_n - f_{n-1}$, $f_0 = 0$ ([1]-[3]).

3. Weak type result. Let $T$ be an operator from $H_p$ to the set of random variables defined on a $\sigma$-finite measure space $(\Omega, \mathcal{F}, \tilde{P})$.

Theorem. Suppose that

1. $T$ is quasi-linear, i.e. $|T(f + g)| \leq C |Tf| + |Tg|$.

2. $\tilde{P}(\{w; |Tf(w)| > t\})^{1/q} \leq M_t/t \| f \|_{p'},$ for all $t > 0$, where $1 \leq p \leq q_i < \infty$ ($i=0,1$), $p_0 \neq p_1$ and $q_0 = q_1$. Let us put $1/p = (1 - \theta)/p_0 + \theta/p$, and $1/q = (1 - \theta)/q_0 + \theta/q_1$, $0 < \theta < 1$. Then

$$\| Tf \|_q \leq AC(C + 1)M_t^{-\theta}M_{t_0}^{\theta} \| f \|_p,$$

where $$A^q = 0((q_1 - q)^{-1} + (q - q_0)^{-1} + q - q_0)$$.

Proof. We consider the case $1 = p_0 < p_1$ and $q_0 < q_1$ only, the other cases are treated similarly.

1-st step. The following decomposition lemma is used in the proof, which corresponds to the Calderón-Zygmund decomposition ([4]-[6]).

Lemma (R. Gundy). Let $v \in L^1(\Omega), r \geq 1$. Then for each $a > 0$, $v$ is decomposed into three random variables $g, h, k, v = g + h + k$, which satisfy

1. $P(\{w; g^*(w) > 0\}) \leq K/a \| v \|,$

2. $\| \sum_{n=1}^\infty |Ah_n| \|_1 \leq K \| v \|,$

where $|Ah_n| = |Ah_n| - |Ah_{n-1}|$, $h_0 = 0$.
with a constant K independent of a, v, r.

Now put \( \lambda = p_0(q - q_0)/q_0(p - p_0), \rho = -q_0/(q_1 - q_0), \sigma = q_1/(q_1 - q_0), \tau = (p_0q_0 - p_0q_1)/p_0(q_0 - q_1), B = M_1 M_0', f = f, r = (p + 1)/2 (> 1). \)

2-nd step. Let \( f \in L^p(\Omega) \). Then for each \( y > 0 \) the following decomposition of \( f \) is possible:

1. \( f = u + u', \ u' = k + g + h \)
2. \( u = f, \text{ if } |f| < (y/B)^t \) and \( u = 0, \text{ elsewhere}. \)
3. There exists a constant K independent of \( y, u', r \), so that
   \[
   \|k\|_{p_1} \leq K(y/B)^{(p_1 - r)} \|u'\|^r, \\
   \|g\|_{l_1} \leq K A_r (y/B)^{(1 - r)} \|u'\|^r \\
   \|h\|_{l_1} \leq K (y/B)^{(1 - r)} \|u'\|^r.
   \]

In fact (3) is shown by lemma as follows. Put \( v = u' \) and \( a = (y/B)^t \) in the following inequalities.

\[
\|k\|_{p_1} \leq \int |k| \, dP \cdot \|k\|_{p_1 - 1} \leq K \|v\|_1 \leq K \int |v|^r \, dP, \\
\|g\|_{l_1} \leq P(g^* > 0)^{1/r} \int (g^*)^r \, dP \leq (K/\alpha \|v\|_1)^{1/r} A_r \|g\|_r \leq K a^{1/r} \cdot A_r \|v\|^{r(r + 1)}
\]

and

\[
\|h\|_{l_1} \leq \sum_{n=1}^{\infty} |A h_n| \leq K \|v\|_1 \leq K a^{1/r} \|v\|_r.
\]

3-rd step. Considering the decomposition above, we may write

\[ \|Tf\|_s = q \int_0^\infty y^{q-1} P(|Tf| > y) \, dy \leq q(4c(c + 1))q(I_1 + I_2 + I_3 + I_4), \]

where

\[ I_1 = \int_0^\infty y^{q-1} P(|Tu| > y) \, dy, \]
\[ I_2 = \int_0^\infty y^{q-1} P(|Tk| > y) \, dy, \]
\[ I_3 = \int_0^\infty y^{q-1} P(|Tg| > y) \, dy, \]
\[ I_4 = \int_0^\infty y^{q-1} P(|Th| > y) \, dy. \]

Now the rest of the proof is almost the same as in [4]. For example, we estimate the value \( I_3 \) as follows.

\[ I_3 \leq M_0^g \int_0^\infty y^{q_0 - 1} \|g\|_{p_0} \, dy \leq K A_{p_0} M_0^g B^{(r - 1)/q_0} \int_0^\infty y^{q_0 - 1 + 1/2} \|u'\|_{r_0} \, dy \leq K A_{p_0} M_0^g B^{(r - 1)/q_0} \left\{ \left\{ \int_0^\infty y^{q_0 - 1 + 1/2} \|u'\|_{r_0} \, dy \right\}^{1/2} \right\}^{q_0} \]
\[\begin{align*}
&\leq KA_0^p M_0^p B^{q-2} \left( (q-q_0) + \lambda (1-r)q_0 \right) \left[ \int |f|^{(q-q_0)/q_0} q_0^{q_0+1} dP \right]^{q_0} \\
&\leq 0(1/(r-1)^{2q} (q-q_0)) M_0^{q(q-2)} M_0^{q} ||f||_p^q \\
\text{Q.E.D.}
\end{align*}\]

4. Remarks.

(1) The result also holds even if \( P(Q) = \infty \).

(2) If \( X = (X_n)_{n=0}^\infty \) is a martingale with respect to \( (Q, \mathcal{F}, P, \{\mathcal{F}_n\}_{n=1}^\infty) \) we say that \( X \in M_p \) \( (1 \leq p < \infty) \) when \( \|X\|_{M_p} = \sup_{n \geq 1} E(|X_n|^p)^{1/p} < \infty \). Then \( H_p \) is isomorphic to \( M_p \) for \( 1 \leq p < \infty \) by the correspondence, \( f(w) \leftrightarrow X_n(w) = \lim_{n \to \infty} X_n(w) \). Therefore it is concluded that the interpolation theorem of operators also holds on martingale spaces \( M_p \) for \( 1 \leq p < \infty \).

References


