77. On the System of Pfaffian Equations of Briot-Bouquet Type

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§ 1. Introduction. In this paper we shall extend some well-known results on the system of ordinary differential equations of Briot-Bouquet type to the system of Pfaffian equations. By a system of Pfaffian equations of Briot-Bouquet type we mean a completely integrable system of Pfaffian equations

\[ du_i = \sum_{k=1}^{n} \frac{f^{ik}(u_1, \ldots, u_m, x_1, \ldots, x_n)}{x_k} \, dx_k, \quad i = 1, \ldots, m, \]

or

\[ x_k \frac{\partial u_i}{\partial x_k} = f^{ik}(u, x), \quad i = 1, \ldots, m; \quad k = 1, \ldots, n, \]

where the \( f^{ik} \) are functions holomorphic at the origin \( u_1 = \cdots = u_m = x_1 = \cdots = x_n = 0 \) and vanishing there. By the use of the usual multi-index notation: \( \alpha = (\alpha_1, \ldots, \alpha_m) \), \( \beta = (\beta_1, \ldots, \beta_n) \), the Taylor expansions of the \( f^{ik} \) are expressible as

\[ f^{ik}(u, x) = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a^{ik}_{\alpha \beta} u_{\alpha} x_{\beta}. \]

By denoting \( A_k \) the matrix formed by the coefficients of \( u_1, \ldots, u_m \) in the developments of \( f^{ik}, \ldots, f^{mk} \), let \( \lambda_1^k, \ldots, \lambda_m^k \) be the eigenvalues of \( A_k \).

The complete integrability condition for (1) can be written as follows:

\[ \sum_{\mu=1}^{m} \frac{\partial f^{il}}{\partial u_{\mu}} f^{\nu k} + x_k \frac{\partial f^{il}}{\partial x_k} = \sum_{\mu=1}^{m} \frac{\partial f^{ik}}{\partial u_{\mu}} f^{\mu l} + x_l \frac{\partial f^{ik}}{\partial x_l}. \]

§ 2. Formal integration.

Theorem 2.1. Suppose that

(i) All the \( A_k, k = 1, \ldots, n \), are similar to diagonal matrices;

(ii) For any system of non-negative integers \( (\alpha_1, \ldots, \alpha_m, B) \), there exists an index \( K, 1 < K < n \), such that

\[ \sum_{\rho=1}^{m} \alpha_{\rho} \lambda^K_{\rho} + B, \quad i = 1, \ldots, m. \]

Then there exists a formal transformation of the form

\[ u_i = \sum_{\rho=1}^{m} p_{i\rho} x_{\rho} + \sum_{\nu=1}^{n} p^{i}_\nu x_{\nu} + \sum_{\nu=1}^{n} p^{i}_\nu p^{\nu}_{\nu} x_{\nu}, \]

which transforms the system (1) into the system
where $P=(p_{ij}) \in \text{GL}(m, \mathbb{C})$ and $\lambda_1^k, \ldots, \lambda_m^k$ are suitably renumbered for each $k$.

**Theorem 2.2.** Suppose that there exists an index $K$, $1 \leq K \leq n$, such that

$$\lambda^K = \sum_{p=1}^{m} \alpha_p \lambda^K_p + B,$$

for any system of non-negative integers $(\alpha_1, \ldots, \alpha_m, B)$ with the exception of the trivial $m$ equalities: $\lambda^K_i = \lambda^K_i$.

Then there exists a formal transformation (3), which transforms the system (1) into the system (4).

In order to prove Theorems 2.1 and 2.2, it is sufficient to prove the following three lemmata:

**Lemma 1.** There exists an invertible linear transformation

$$u_i = \sum_{s=1}^{m} p_{is} v_s,$$

which takes (1) into a system

$$x_k \frac{\partial v_i}{\partial x_k} = \lambda_i^k v_i + \sum_{s=1}^{m} b_{is}^k x_s + \sum_{|\alpha|+|\beta| \geq 2} b_{\alpha\beta}^k v^\alpha x^\beta.$$

**Lemma 2.** For a completely integrable system

$$x_k \frac{\partial u_i}{\partial x_k} = \lambda_i^k u_i + \sum_{s=1}^{m} a_{is}^k x_s + \sum_{|\alpha|+|\beta| \geq 2} a_{\alpha\beta}^k u^\alpha x^\beta,$$

one can find a unique transformation

$$u_i = v_i + \sum_{s=1}^{m} p_{is} v_s,$$

which transforms (5) into a system

$$x_k \frac{\partial v_i}{\partial x_k} = \lambda_i^k v_i + \sum_{|\alpha|+|\beta| \geq 2} b_{\alpha\beta}^k v^\alpha x^\beta.$$

**Lemma 3.** A completely integrable system of the form

$$x_k \frac{\partial u_i}{\partial x_k} = \lambda_i^k u_i + \sum_{|\alpha|+|\beta| \geq N} a_{\alpha\beta}^k u^\alpha x^\beta,$$

$N \geq 2$,

is transformed by a transformation and only one

$$u_i = v_i + \sum_{|\alpha|+|\beta| = N} p_{\alpha\beta}^k v^\alpha x^\beta$$

into a system

$$x_k \frac{\partial v_i}{\partial x_k} = \lambda_i^k v_i + \sum_{|\alpha|+|\beta| \geq N+1} b_{\alpha\beta}^k v^\alpha x^\beta.$$

Lemma 1 is an immediate consequence of the assumption (i) of Theorem 2.1 or the assumption of Theorem 2.2 and the relations $A_k A_i = A_i A_k$ which are deduced from (2). Lemma 2 is easily proved from the assumption (ii) of Theorem 2.1 or the assumption of Theorem 2.2 and the relations.
which are derived from the complete integrability condition for (5). From the integrability condition for (6) we obtain

$$
(\sum_{\nu=1}^{m}(\alpha_{\nu}-\delta_{\nu})\chi_{\nu}^{\nu}+\beta_{\nu})a_{\nu}^{\nu}= \left(\sum_{\nu=1}^{m}(\alpha_{\nu}-\delta_{\nu})\chi_{\nu}^{\nu}+\beta_{\nu}\right)a_{\nu}^{\nu}.
$$

The transformation (7) is invertible as

$$
v_{i} = u_{i} - \sum_{|\alpha|+|\beta|\geq 2} p_{\alpha\beta} u_{\alpha} x^{\beta} + \cdots,
$$

whence

$$
x_{k} \frac{\partial v_{i}}{\partial x_{k}} = \chi_{k}^{i} u_{i} + \sum_{|\alpha|+|\beta|\geq 2} \left( a_{\alpha\beta}^{\nu} - \left( \sum_{\nu=1}^{m}(\alpha_{\nu}-\delta_{\nu})\chi_{\nu}^{\nu}+\beta_{\nu}\right) p_{\alpha\beta}^{\nu}\right) u_{\alpha} x^{\beta} + \cdots.
$$

Inserting (7) into the right-hand side,

$$
x_{k} \frac{\partial v_{i}}{\partial x_{k}} = \chi_{k}^{i} v_{i} + \sum_{|\alpha|+|\beta|\geq 2} \left( a_{\alpha\beta}^{\nu} - \left( \sum_{\nu=1}^{m}(\alpha_{\nu}-\delta_{\nu})\chi_{\nu}^{\nu}+\beta_{\nu}\right) p_{\alpha\beta}^{\nu}\right) u_{\alpha} x^{\beta} + \cdots,
$$

from which follows Lemma 3 in virtue of (8).

§ 3. Convergence of formal transformation.

Theorem 3.1. Suppose that the assumptions (i), (ii) of Theorem 2.1 and the following assumption are verified:

(iii) For each $k$, $k=1, \cdots, n$, one finds, in the complex plane, a straight line passing through the origin in such a way that the eigenvalues $\lambda_{1}^{k}, \cdots, \lambda_{m}^{k}$ and unity lie in the same side of the line.

Then the formal transformation (3) does converge.

Theorem 3.2. The formal transformation (3) converges under the assumption of Theorem 2.2 and the following:

(iii)' The eigenvalues $\lambda_{1}^{k}, \cdots, \lambda_{m}^{k}$ and 1 lie in the same side of a straight line in the complex plane passing through the origin.

There is no loss of generality in supposing that the system (1) is of the form

$$
x_{k} \frac{\partial u_{i}}{\partial x_{k}} - \chi_{k}^{i} u_{i} = \sum_{|\alpha|+|\beta|\geq 2} a_{\alpha\beta}^{\nu} u_{\alpha} x^{\beta}.
$$

Then the formal transformation (3) takes the following form:

$$
u_{i} = v_{i} + \sum_{|\alpha|+|\beta|\geq 2} p_{\alpha\beta}^{\nu} v_{\alpha} x^{\beta}.
$$

Substituting (10) into (9) and using (4), we obtain

$$
\left( \sum_{\nu=1}^{m}(\alpha_{\nu}-\delta_{\nu})\chi_{\nu}^{\nu}+\beta_{\nu}\right) p_{\alpha\beta}^{\nu} = P_{\alpha\beta}(p_{\alpha\beta}^{\nu}, a_{\alpha\beta}^{\nu}),
$$

where the $P_{\alpha\beta}$ are polynomials in $p_{\alpha\beta}^{\nu}, 1 \leq i \leq m, |\alpha'|+|\beta'| < |\alpha|+|\beta|$, whose coefficients are linear forms in $a_{\alpha\beta}^{\nu}, |\alpha''|+|\beta''| \leq |\alpha|+|\beta|$. We take a convergent power series $\sum_{|\alpha|+|\beta|\geq 2} A_{\alpha\beta} u_{\alpha} x^{\beta}$, which is a majorizing series for all $\sum_{|\alpha|+|\beta|\geq 2} a_{\alpha\beta}^{\nu} u_{\alpha} x^{\beta}$, and set

$$
F(u, x) = \sum_{|\alpha|+|\beta|\geq 2} A_{\alpha\beta} u_{\alpha} x^{\beta}.
$$

Next we choose a positive constant $\rho$ so that we have

$$
\left| \sum_{\nu=1}^{m}(\alpha_{\nu}-\delta_{\nu})\chi_{\nu}^{\nu}+\beta_{\nu}\right| \geq \rho
$$

which is a contradiction, hence the convergence of the formal transformation (3) is proved.
for some $K, 1 \leq K \leq n$, and for any $(\alpha, \beta)$ with $|\alpha| + |\beta| \geq 2$. We see that the system of equations in $u_1, \ldots, u_m$
\[ \rho(u_i - v_i) = F(u, x) \]
has a solution expressible by convergent series
\[ u_i = v_i + \sum_{|\alpha| + |\beta| \geq 2} P_{\alpha, \beta} v^{\alpha} x^{\beta} \]
and that $\sum_{|\alpha| + |\beta| \geq 2} P_{\alpha, \beta} v^{\alpha} x^{\beta}$ is a majorizing series of $\sum_{|\alpha| + |\beta| \geq 2} p_{\alpha, \beta} v^{\alpha} x^{\beta}$ for $i = 1, \ldots, m$.

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Reference