91. *Non-Equilibrium Behavior of Turbulent Boundary Layer Flows. I*

Method of Analysis

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Introduction. For a two-dimensional incompressible turbulent boundary layer on a smooth wall, we have to deal with two different length scales, a small viscous scale ν/u_τ and a much larger, boundary layer thickness δ , where ν is the kinematic viscosity and u_τ is the friction velocity. In terms of these length scales, the mean velocity profile is postulated in the form

$$(1) \quad \frac{U}{u_\tau} = F\left(\frac{u_\tau y}{\nu}, \frac{y}{\delta}\right),$$

where U is the mean velocity in the x -direction, x and y being coordinates along and normal to the solid wall, respectively. It is by virtue of the celebrated work of Coles (1956) that has provided the postulate (1) with the form of the additive law of the wall and law of the wake

$$(2) \quad \frac{U}{u_\tau} = \frac{1}{\kappa} \ln \frac{u_\tau y}{\nu} + C + \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right)$$

for the region outside the viscous sublayer ($y > 50 \nu/u_\tau$, say), where κ is the Kármán constant, C is the smooth-wall constant, Π is the wake parameter, and w is the universal function of y/δ . There is now abundant evidence to believe that the conditions under which the equation (2) is valid are nearly equivalent to those under which the boundary layer approximation itself might be expected to apply.

The wake parameter Π is generally regarded as dependent on x . In the event that Π is independent of x , the velocity defect relative to the free-stream velocity U_0 , made non-dimensional by u_τ , turns out to be function of y/δ only, which is the property assigned to equilibrium flow by Clauser (1954). The existence of such an equilibrium boundary layer has been observed experimentally for a certain range of Reynolds numbers both in zero and positive pressure gradients (Clauser 1954; Bradshaw 1967). With the equilibrium flow as a datum, the perturbed boundary layer could be described in terms of approach to, or departure from equilibrium (Coles 1962, 1969). From the viewpoint of dimensional arguments, however, the equilibrium turbulent boundary layer could only be reconciled with the Reynolds averaged momentum equation if u_τ/U_0 and $d\delta/dx$ were independent of x , namely, for example, if the flat plate were roughened in such a way that the roughness scale increases linearly with the distance from the leading edge (Rotta 1962). The conditions are not fulfilled with flat plate boundary layers having smooth or uniformly rough walls. The almost equilibrium as observed experimentally thus raises a question as to why u_τ/U_0 and $d\delta/dx$ exert such a weak influence.

In an extensive survey of experimental data of various boundary layer flows, Coles (1962, 1969) evaluated the wake parameter Π by fitting the measured

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velocity profile to the analytical form (2), and classified the boundary layer behavior by the variation of Π in the x -direction. The results obtained are exceedingly illuminating in a wide variety of respects. To mention only a few, the values of the universal constants $\kappa=0.41$ and $C=5.0$ are proposed; the difference between the normal and abnormal behaviors in relaxation is discerned; the tendency for Π to diminish at low Reynolds numbers is suggested. To be critical, however, the procedure of fitting calls for some skill, and the question of equilibrium flow still remains unsettled.

An attempt is made in the present and succeeding papers to put forward a simple method of analysis for evaluating characteristic parameters from experimental mean velocity data (Part I), and set forth the results of analysis for throwing additional light upon the equilibrium or non-equilibrium behavior of turbulent boundary layer flows (Part II).

Integral method of analysis. The universal wake function suggested by Coles (1969) is $w(y/\delta)=1-\cos(\pi y/\delta)$, which, however, does not yield zero velocity gradient at $y=\delta$. The deficiency is made up by using a quartic polynomial (Lewkowicz 1982)

$$(3) \quad w\left(\frac{y}{\delta}\right) = 2 \frac{y^2}{\delta^2} \left(3 - 2 \frac{y}{\delta}\right) - \frac{1}{\Pi} \frac{y^2}{\delta^2} \left(1 - \frac{y}{\delta}\right) \left(1 - 2 \frac{y}{\delta}\right),$$

in which the meaning of the universal function is modified, with the second term complementarily correcting the law of the wall. A quartic is chosen in preference to a cubic by reason of the smallness in magnitude of the second term. Evaluating the velocity profile (2) at $y=\delta$ gives

$$(4) \quad \delta = \frac{\nu}{U_0} \frac{E}{\kappa} z,$$

where

$$(5) \quad z = \kappa \frac{U_0}{u_\tau} = \kappa \sqrt{\frac{2}{c_f}}, \quad E = \exp(z - 2\Pi - \kappa C),$$

while using (2) and (3) results in the displacement and momentum thicknesses

$$(6) \quad \delta^* = \frac{\nu}{U_0} \frac{E}{\kappa} \alpha, \quad \theta = \frac{\nu}{U_0} \frac{E}{\kappa} \left(\alpha - \frac{\beta}{z}\right),$$

respectively, where $c_f = 2u_\tau^2/U_0^2$ is the friction coefficient and

$$(7) \quad \alpha = \frac{59}{60} + \Pi, \quad \beta = \frac{8437}{4200} + \frac{667}{210} \Pi + \frac{52}{35} \Pi^2.$$

In calculating δ^* and θ , the Reynolds number is assumed to be high enough for the contributions from the viscous sublayer is negligible. This is not the case for some experimental data, and a correction is needed for the effect of the viscous sublayer. Using the sublayer velocity profile (Spalding 1961)

$$(8) \quad \frac{u_\tau y}{\nu} = \frac{\zeta}{\kappa} + e^{-\kappa C} \left(e^\zeta - 1 - \zeta - \frac{1}{2} \zeta^2 - \frac{1}{6} \zeta^3 - \frac{1}{24} \zeta^4 \right),$$

where $\zeta = \kappa U/u_\tau$, and taking the values $\kappa=0.41$ and $C=5.0$, we obtain

$$(9) \quad \delta^* = \frac{\nu}{U_0} \left(\frac{E}{\kappa} \alpha + 50.63 \right), \quad \theta = \frac{\nu}{U_0} \left\{ \frac{E}{\kappa} \left(\alpha - \frac{\beta}{z} \right) - \left(50.63 - \frac{136.47}{\kappa z} \right) \right\},$$

instead of (6). Writing

$$(10) \quad \delta_c^* = \delta^* - 50.63 \frac{\nu}{U_0}, \quad \theta_c = \theta + \left(50.63 - \frac{136.47}{\kappa z} \right) \frac{\nu}{U_0},$$

(9) may be put in the form

$$(11) \quad \delta_c^* = \frac{\nu}{U_0} \frac{E}{\kappa} \alpha, \quad \theta_c = \frac{\nu}{U_0} \frac{E}{\kappa} \left(\alpha - \frac{\beta}{z} \right),$$

which is the same as (6). It is thus possible to account for the sublayer effect by simply using the corrected thicknesses δ_c^* and θ_c in place of the measured thicknesses δ^* and θ . Eliminating $\nu E/\kappa U_0$ between the two equations of (11) and using (7), we have

$$(12) \quad \frac{8437}{4200} - \frac{59}{60} z \left(1 - \frac{\theta_c}{\delta_c^*}\right) + \left\{ \frac{667}{210} - z \left(1 - \frac{\theta_c}{\delta_c^*}\right) \right\} \Pi + \frac{52}{35} \Pi^2 = 0.$$

On the other hand, elimination of E between the second equation of (5) and the first equation of (11) yields

$$(13) \quad \frac{\kappa}{\alpha} \frac{U_0 \delta_c^*}{\nu} = \exp(z - 2\Pi - \kappa C).$$

These equations afford the means for evaluating z and Π from the measurable quantities δ^* and θ .

In the following, we confine ourselves to the boundary layer on a smooth wall in zero pressure gradient, where U_0 is independent of x . With the momentum thickness θ as given by (9), the momentum integral equation $d\theta/dx = u_\tau^2/U_0^2$ leads to

$$(14) \quad e^{-\sigma} \frac{d}{d\sigma} \left\{ \frac{E}{\kappa} \left(\alpha - \frac{\beta}{z} \right) - \left(50.63 - \frac{136.47}{\kappa z} \right) \right\} = \frac{\kappa^2}{z^2},$$

where $\sigma = \ln(U_0 x/\nu)$, or

$$(15) \quad (P - \Delta) \frac{dz}{d\sigma} - Q \frac{d\Pi}{d\sigma} - \frac{\kappa^3 e^\sigma}{E} = 0,$$

with the abbreviating notations

$$(16) \quad P = \alpha z^2 - \beta(z-1), \quad Q = \left(2\alpha - \frac{d\alpha}{d\Pi} \right) z^2 - \left(2\beta - \frac{d\beta}{d\Pi} \right) z,$$

$$(17) \quad \Delta = \frac{136.47}{E}.$$

Friction and wake parameters. As mentioned above, the equations (12) and (13) provide the means for evaluating the friction parameter z and the wake parameter Π from the experimental values of δ^* and θ . If, in addition, z could be estimated from experiment, it would seem most reasonable to adjust

Fig. 1

Fig. 2

the values of δ^* , θ and z in such a way that (12) and (13) are satisfied. Instead, however, we simply assume that δ^* and θ are correct but z is susceptible to error and to be adjusted such that these equations be satisfied. This procedure appears to be justified on the ground that most of the experimental data tabulated by Coles (1969) are made consistent by adjustment of z by less than 2 to 3 percent.

It is also to be noted that the effect of viscous sublayer is confined to a single term Δ in (15). Inclusion of this term is necessary unless the Reynolds number $U_0 x/\nu$ is much in excess of 10^6 ($\sigma=14$).

The method of evaluating z and Π , though different in formulation, appears to be nearly equivalent in principle to that used by Coles (1969). As a matter of fact, calculated values of z and Π are in good agreement with those given by Coles, as will be shown below. When viewed as a method of estimating the friction parameter z from the mean velocity profile, the present procedure is simpler compared to the method of Kapinos and Tarasov (1980), in which the effect of the wake component of velocity profile is also taken into account. It is to be remembered, however, that the application of the analysis is not confined to the evaluation of z and Π by means of (12) and (13). The equation (15) originating from the momentum integral equation provides a differential equation for z and Π as functions of σ . In case when z is known in terms of σ , (15) can be integrated to obtain Π by the method of isoclines.

In Figs. 1 and 2 are shown the values of z and Π for given values of σ , calculated by (12) and (13) from the mean velocity data of a turbulent boundary layer in zero pressure gradient, originally due to Wieghardt (1943) and tabulated by Coles (1969). Owing to the adjustment, values of z are slightly different from those tabulated, the difference being less than 1 percent. The difference for Π is more marked, but this is partly due to the difference in definition. With the curve of z as drawn in Fig. 1, some of the isoclines and integral curves are shown in Fig. 2 to be compared with the calculated values of Π . In view of the accuracy of determining Π , the result of comparison might be considered as fairly satisfactory.

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