26. Equilibrium, or Nonequilibrium, of Turbulent Boundary Layer Flows

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Introduction. A two-dimensional turbulent boundary layer is called self-preserving, or in equilibrium, when the mean velocity defect profile outside the viscous sublayer \( y > 50 \nu/\delta \) retains the form \( U_0 - U = u_F(y/\delta) \) in the course of its downstream development, where \( U \) is the mean velocity in x-direction, \( x \) and \( y \) are coordinates parallel and normal to the wall, respectively, \( U_0 \) is the free-stream velocity, \( \nu \) is the friction velocity, \( \nu \) is the kinematic viscosity, and \( \delta \) is the thickness of the boundary layer. The existence of flow bearing such a feature has been observed experimentally for certain ranges of Reynolds number and pressure gradient (Ludwieg and Tillmann 1949; Clauser 1954; Herring and Norbury 1967; Bradshaw 1967; East and Sawyer 1980). It is also well known that the mean velocity profile outside the viscous sublayer is expressed with sufficient accuracy by the combined law of the wall and law of the wake due to Coles (1956)

\[
\frac{U}{\nu} = \frac{1}{k} \ln \frac{u_F y}{\nu} + C + \frac{\Pi}{k} w - \frac{\Delta U}{u_F},
\]

where \( k \) is the Karman constant \((=0.41)\), \( C \) is the smooth-wall constant \((=5.0)\), \( \Pi \) is the wake parameter, \( w \) is the universal function of \( y/\delta \), and \( \Delta U/u_F \) is the constant shift due to wall roughness. With the assumption of (1), existence of equilibrium is characterized by the condition that \( \Pi \) is independent of \( x \).

On dimensional grounds, however, the equilibrium boundary layer could only be reconciled with the momentum equation provided that \( u_F \) and \( d\delta/dx \) were independent of \( x \) (Rotta 1962). These conditions being not fulfilled with smooth-wall boundary layers in zero pressure gradient, the equilibrium observed in experiments should be no more than an approximate one. This paper advances the author's view that strict discrimination should be made between equilibrium and nonequilibrium behaviors of turbulent boundary layer.

Characteristic parameters: evaluation. Substituting the quartic approximation due to Lewkowicz (1982),

\[
w = 2 \frac{y^2}{\delta^2} \left( 3 - 2 \frac{y}{\delta} \right) - \frac{1}{\Pi} \frac{y^2}{\delta^2} \left( 1 - \frac{y}{\delta} \right) \left( 1 - 2 \frac{y}{\delta} \right),
\]

to the wake function \( w \) into the mean velocity profile (1) and integrating, we obtain the displacement and momentum thicknesses of the boundary layer

\[
\delta^* = \alpha \frac{\delta}{z}, \quad \theta = \left( \alpha - \frac{\beta}{z} \right) \frac{\delta}{z},
\]

respectively, where

\[
z = \kappa \frac{U_0}{u_F} = \kappa \sqrt{\frac{2}{c_f}},
\]
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\[ \alpha = \frac{59}{60} + \Pi, \quad \beta = \frac{8437}{4200} + \frac{667}{210} \Pi + \frac{52}{35} \Pi^2, \]

c_{f} being the conventional skin friction coefficient. Furthermore, we have

\[ \delta = \frac{\nu}{U_0} \frac{z}{k} e^s \]

for smooth-wall boundary layer \((\Delta U/u_r = 0)\),

\[ \delta = h e^{s + \kappa K} \]

for k-type rough-wall boundary layer \((\Delta U/u_r = \kappa^{-1} \ln (u_r/\nu) + K)\), and

\[ s = -\kappa D \]

for d-type rough-wall boundary layer \((\Delta U/u_r = \kappa^{-1} \ln (u_r/\nu) + D)\), respectively, where \(s = z - 2\Pi - \kappa C\), \(h\) is the scale of roughness and \(K\) and \(D\) are constants.

For low Reynolds numbers the contributions from the viscous sublayer close to the smooth wall are required for correction to the expressions of (3). Equations (3) to (8) afford the means for evaluating the characteristic parameters \(z\) and \(\theta\) when \(K\) and \(D\) are known from other sources. When \(K\) and \(D\) are also to be known, the momentum integral equation (next section) must be used as an auxiliary equation for evaluation.

This method of analysis was applied first to the measured velocity profiles of adequately tripped boundary layer on a smooth flat plate in zero pressure gradient (Tani and Motohashi 1985). Result of evaluation appears to have denied the existence of equilibrium as characterized by \(\Pi = \text{constant}\), a conclusion at variance with that commonly believed.

**Characteristic parameters: downstream development.** With \(\delta^o\) and \(\theta\) as given by (3), (6), (7) and (8), the momentum integral equation

\[ \frac{d\theta}{dx} + \frac{\delta^o + 2\theta}{U_0} \frac{dU_0}{dx} = \frac{u_r^2}{U_0^2} \]

leads to

\[ \begin{align*}
(az - \beta z) \frac{dz}{dx} - [(2z - \alpha)z - (2\beta z - \dot{\beta})] \frac{d\Pi}{dx} \\
= \dot{\Pi} + (2az - \beta z) \frac{dU_0}{dx} = \frac{U_0}{\nu} k^o e^{-\kappa},
\end{align*} \]

\[ + (az - \beta) \frac{dh}{dx} + (3az - 2\beta) \frac{h}{U_0} \frac{dU_0}{dx} = \kappa^2 e^{(s + \kappa K)}, \]

\[ \begin{align*}
(az - (2az + \beta) + \frac{2\beta z}{z}) \frac{d\delta}{dx} + (az - \beta) \frac{d\Pi}{dx} \\
+ \frac{4\beta}{z} \frac{\delta}{2} \frac{dz}{dx} + (az - \beta) \frac{d\delta}{dx} \\
+ (3az - 2\beta) \frac{\delta}{U_0} \frac{dU_0}{dx} = \kappa^2, \end{align*} \]

for smooth-wall, k-type rough-wall and d-type rough-wall boundary layers, respectively, where the dot denotes differentiation with respect to \(\Pi\), and \(\dot{\beta} = \beta - 136.47e^{-s}\), containing the correction due to viscous sublayer.
Each of these equations, though incapable of solution without being supplemented by an auxiliary equation, serves to examine the existence of equilibrium solution. With $dU_0/dx = 0$ and $dH/dx = 0$, for example, (10) is integrated to give $z$ as function of $U_0/\nu$, which is shown by solid lines in Fig. 1, to be compared with the evaluation (preceding section) from the experimental data of adequately tripped smooth-wall boundary layer in zero pressure gradient. The difference between theoretical and experimental results is rather small, but the former is represented by a slightly curved line, while the latter by a dotted straight line. This kind of disparity should not be overlooked, since the question posed is as to whether $H=\text{constant}$ is a solution to the equation (10). We thus conclude that there exists no equilibrium, reinforcing the argument advanced in the preceding section.

![Fig. 1](image-url)

**Possibility of equilibrium: accelerating flow.** Seeing that the viscosity-dependent friction $(dz/dx \neq 0)$ precludes the possibility of equilibrium for smooth-wall boundary layer in zero pressure gradient, we seek for equilibrium flow with the condition that $dz/dx = 0$ and $dH/dx = 0$ (Tani 1986). It is then readily seen from (10) and (11) that equilibrium would be possible only for accelerating flow $(dU_0/dx > 0$, favorable pressure gradient) over smooth as well as $k$-type rough walls. If the roughness is allowed to increase in height linearly with $x$, equilibrium would be possible even for uniform flow (zero pressure gradient). These possibilities having been anticipated by Rotta (1962) on dimensional grounds, it seems desirable to corroborate the existence of equilibrium on the basis of evaluation from available experimental data.

Values of $z$ and $H$ calculated from measured velocity profiles of decelerating smooth-wall (Clauser 1954) and accelerating smooth-wall (Herring and Norbury 1967) flows are shown in Figs. 2 and 3, respectively. When viewed in perspective, $z$ and $H$ are kept nearly constant for both flows, at least in sufficiently downstream regions. For decelerating flow, however, the pressure gradient parameter, $A = - (U_0/\nu)^2 (U_0 \partial \beta / \nu) [\nu (U_0^3) (dU_0/dx)]$, takes a positive value, which cannot be reconciled with the requirement deduced from (10), namely

$$ [(2\alpha z - \beta)e^t + 136.47] \frac{z}{U_0} \frac{dU_0}{dx} = \frac{U_0}{\nu} \kappa$. $$

For accelerating flow, the condition (13) is fulfilled by the velocity distribution $U_0 = \kappa^{-3}z[(2\alpha z - \beta)e^t + 136.47] (x_0 - x)^{-1}$, which would be realized by the flow between converging plane walls. Unfortunately, however, most of the existing experiments were made in lower pressure gradients, without regard to the requirement (13). As a natural result, the corroboration of equilibrium has re-
mained not entirely satisfactory.\(^9\)"

**Possibility of equilibrium: d-type roughness.** The d-type roughness, as typified by the wall containing sparsely spaced spanwise grooves, is characterized by the roughness shift of the form independent of roughness scale \((dU/u_z = x^{-1}\ln(u_z\delta/\nu) + D)\). For d-type rough-wall boundary layers (12) predicts the equilibrium solution, \(dz/dx = 0\) and \(d\Pi/dx = 0\), with \((\alpha z - \beta)(d\delta/dx) = \kappa^2 - (3\alpha z - 2\beta)\times(\partial/\partial_x(U)_0)(dU_0/dx)\), although the possibility has come into notice since the experiment due to Perry, Schofield and Joubert (1969). It is expected that equilibrium would exist for a certain range of pressure gradients, but the available experiments are confined to zero pressure gradient (Perry et al. 1969; Wood and Antonia 1975; Osaka, Nakamura and Kageyama 1984; Matsumoto, Munakata and Abe 1986). In Fig. 4 are shown the results of evaluation from the measurements of Matsumoto et al. on a flat plate roughened with spanwise grooves of 5 mm depth and 5 mm width, repeating at a spacing of 50 mm, and beginning at 0.4 m from the leading edge. It is seen that both \(z\) and \(\Pi\) are nearly constant in \(x\)-direction and that \(\delta\) increases almost linearly with \(x\). The evaluation thus provides convincing evidence in support of the theoretical prediction.

**Recovery to equilibrium: d-type roughness.** Recently, additional evidence equilibrium has been furnished by perturbation experiments, carried out by Matsumoto, Munakata and Abe (1986) on the author's request. The turbulent boundary layer over a d-type rough flat plate was perturbed by a spanwise rod of small diameter \(d\), attached to the mid-crest of roughness at a distance of 1 m from the leading edge. Fig. 5 shows results of evaluation from measured velocity profiles. It is seen that both \(z\) and \(\Pi\) increase by perturbation, but tend to approach new equilibrium values, different from those of the unperturbed flow. This is the kind of behavior as might naturally be expected from (12). On the other hand, the like behavior cannot be noted in similarly perturbed

\(^9\) The interpretation previously given on the experimental results of accelerating flow (Tani 1986) is inadequate and should be revised as stated above.
smooth-wall boundary layer flows. As shown in Fig. 6, the effect of perturbation appears to persist for an exceedingly long distance downstream without any sign of recovery. Since tendency of recovery is implicit in the definition of equilibrium, the behavior exhibited by smooth-wall boundary layer is regarded as a strong evidence against the existence of equilibrium.

Concluding remarks: In the preceding sections, emphasis is laid on the consistent determination of \( z \) and \( II \) from measured velocity profiles as well as the corroboration for or against the existence of equilibrium in the context of these parameters. For smooth-wall turbulent boundary layer in zero pressure gradient, the corroboration depends on such a small disparity as indicated in Fig. 1 that it might have looked somewhat perfunctory. As a matter of fact, the skin friction of adequately tripped boundary layer could be calculated with sufficient accuracy by assuming a constant value for \( II \). So far as this kind of problem is concerned, the result seems to be almost insensitive to the assumption of equilibrium or nonequilibrium. When the perturbed boundary layer is to be discussed, however, the relaxation behavior is substantially different according as the unperturbed flow is in equilibrium or not, as observed from the aftereffect of a spanwise rod. In order to make a reliable prediction of the perturbed boundary layer flows, it would appear absolutely necessary to take these situations into account. As a natural consequence, it would be inconsistent to postulate an auxiliary equation which is based on the assumption of equilibrium for predicting the behavior of smooth-wall boundary layer flows.

References