1. *Spiral Vector Method and Symmetrical Component Method*

By Sakae YAMAMURA, M. J. A.

(Communicated Jan. 14, 1991)

Abstract: Expression of state variables plays an important role in analyses and theories. The spiral vector method was proposed by the author as a new way of expressing state variables of electric circuits and electric machinery. It succeeded to unify steady state theory and transient state theory of AC circuits. It was also successful in analyzing transient phenomena of three-phase machines and in deriving superior control of AC motors. Here the method is combined with the symmetrical component method, in order to extend its analytical power to analyses of asymmetrical operation of three-phase machines, which have been left unanalyzed.

Key words: Spiral vector; symmetrical-component transient analysis of AC circuit and AC machinery; asymmetrical operation of three-phase machine.

1. Introduction. The spiral vector can express almost all kinds of state variables, which appear in electrical engineering, and is the most natural form of solutions of circuit equations of electric circuits and performance equations of electrical machines.¹⁾ Both steady and transient state solutions can be expressed by spiral vectors. When spiral vector expressions are used, the steady state theory and transient state theory of the AC circuit, which have been separated because of different expressions of state variables, are unified. Both steady state and transient state analyses of AC machines became simpler and easier. The analytical results thus obtained have provided the theoretical ground on which modern control of AC motors are based. Field Acceleration Method of control of induction motor torque, for example, has thus been developed.¹⁾

The spiral vector method has so far covered analyses of three-phase circuits and machines under symmetrical operation. In analyses of asymmetrical operations of three phase machines and systems the symmetrical component method has been extensively resorted to. However, its analytical power is limited only to steady state analyses, because this method makes use of the phasor expression (vector notation), which can express only steady state AC. S. Bekku²⁾ proposed the instantaneous-value symmetrical component method, in order to extend applicability of the symmetrical component method to transient-state analysis of asymmetrical operation of the three-phase system, but it was not successful. Thus this area has remained unanalyzed. The spiral vector method will truly extend applicability of the symmetrical component method to the unanalyzed area of asymmetrical operation of the three-phase system.

2. Spiral vector. Spiral vector of current is given by

$$(1) \quad i = Ae^{\delta t}, \quad \delta = -\lambda + j\omega$$

As time t goes on, it depicts a spiral on the complex plane in Fig. 1. A general circuit equation is given by

$$(2) \quad i = \frac{A(p)}{B(p)}v$$

Fig. 1. Spiral vector on a complex plane.

Fig. 2. Induction motor model.

, where $A(p)$ and $B(p)$ are polynomials of $p=d/dt$. When forcing function v is an AC voltage given by

$$(3) \quad v = \sqrt{2} |\dot{V}| e^{j(\omega t + \varphi)} = \sqrt{2} \dot{V}$$

, which is a circular vector, the general solution is given by

$$(4) \quad i = \frac{A(j\omega)}{B(j\omega)} v + A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} + \dots + A_n e^{\delta_n t}$$

The first term is the steady state solution and other terms are transient terms, $\delta_1, \delta_2, \dots, \delta_n$ being characteristic roots of $B(p)=0$. Here all terms are spiral vectors. Thus the steady state theory and the transient phenomenon theory of the AC circuit are unified.

3. Instantaneous-real-value symmetrical component method. This method was proposed by S. Bekku.²⁾ It will be now briefly explained. Asymmetrical three phase currents i_a, i_b and i_c are given by the following instantaneous-real-value expressions.

$$(5) \quad \begin{cases} i_a = \sqrt{2} I_0 \cos(\omega t + \varphi_0) + \sqrt{2} I_1 \cos(\omega t + \varphi_1) + \sqrt{2} I_2 \cos(\omega t + \varphi_2) \\ i_b = \sqrt{2} I_0 \cos(\omega t + \varphi_0) + \sqrt{2} I_1 \cos\left(\omega t + \varphi_1 - \frac{2\pi}{3}\right) + \sqrt{2} I_2 \cos\left(\omega t + \varphi_2 + \frac{2\pi}{3}\right) \\ i_c = \sqrt{2} I_0 \cos(\omega t + \varphi_0) + \sqrt{2} I_1 \cos\left(\omega t + \varphi_1 + \frac{2\pi}{3}\right) + \sqrt{2} I_2 \cos\left(\omega t + \varphi_2 - \frac{2\pi}{3}\right) \end{cases}$$

Here I_0 is zero-sequence component, I_1 is positive-sequence component, and I_2 is negative-sequence component. The symmetrical-component-method variable-transformation of eq. (6) is applied to eq. (5), then we get the following symmetrical components in eq. (7).

$$(6) \quad \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \frac{A}{3} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}, \quad \begin{aligned} \alpha &= -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ \alpha^2 &= -\frac{1}{2} + j\frac{\sqrt{3}}{2}, \quad \alpha^3 = 1 \end{aligned}$$

$$(7) \quad \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} I_0 \cos(\omega t + \varphi_n) \\ \frac{1}{\sqrt{2}} I_1 e^{j(\omega t + \varphi_1)} + \frac{1}{\sqrt{2}} I_2 e^{-j(\omega t + \varphi_2)} \\ \frac{1}{\sqrt{2}} I_1 e^{-j(\omega t + \varphi_1)} + \frac{1}{\sqrt{2}} I_2 e^{j(\omega t + \varphi_2)} \end{bmatrix}$$

Zero-sequence component i_0 is real, as it was originally, and positive-sequence component i_1 and negative-sequence component i_2 become exponential time functions, which contain both original positive- and negative-sequence components I_1 and I_2 . The variable transformation has failed to normalize the currents. In this situation it is difficult to proceed with analysis of asymmetrical operation of three-phase systems, and thus this method was not successful.

4. Spiral-vector symmetrical-component method. When asymmetrical three-phase currents of eq. (5) are expressed by circular vectors, eq. (8) is obtained.

$$(8) \quad \begin{cases} i_a = \sqrt{2} |\dot{I}_0| e^{j(\omega t + \varphi_0)} + \sqrt{2} |\dot{I}_1| e^{j(\omega t + \varphi_1)} + \sqrt{2} |\dot{I}_2| e^{j(\omega t + \varphi_2)} \\ i_b = \sqrt{2} |\dot{I}_0| e^{j(\omega t + \varphi_0)} + \sqrt{2} |\dot{I}_1| e^{j(\omega t + \varphi_1 - 2\pi/3)} + \sqrt{2} |\dot{I}_2| e^{j(\omega t + \varphi_2 + 2\pi/3)} \\ i_c = \sqrt{2} |\dot{I}_0| e^{j(\omega t + \varphi_0)} + \sqrt{2} |\dot{I}_1| e^{j(\omega t + \varphi_1 + 2\pi/3)} + \sqrt{2} |\dot{I}_2| e^{j(\omega t + \varphi_2 - 2\pi/3)} \end{cases}$$

Making symmetrical component transformation of eq. (6), the following components are obtained.

$$(9) \quad \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \begin{bmatrix} \sqrt{2} |\dot{I}_0| e^{j(\omega t + \varphi_0)} \\ \sqrt{2} |\dot{I}_1| e^{j(\omega t + \varphi_1)} \\ \sqrt{2} |\dot{I}_2| e^{j(\omega t + \varphi_2)} \end{bmatrix}$$

Each symmetrical component $\dot{I}_0$, $\dot{I}_1$, or $\dot{I}_2$ is separated from others, as was expected from the symmetrical component method. Here normalization of variables has been achieved.

For transient state analysis circular vectors become spiral vectors, and eqs. (8) and (9) become eqs. (10) and (11) respectively

$$(10) \quad \begin{cases} i_a = A_0 e^{\delta_0 t} + A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} \\ i_b = A_0 e^{\delta_0 t} + A_1 e^{\delta_1 t - j2\pi/3} + A_2 e^{\delta_2 t + j2\pi/3} \\ i_c = A_0 e^{\delta_0 t} + A_1 e^{\delta_1 t + j2\pi/3} + A_2 e^{\delta_2 t - j2\pi/3} \end{cases} \quad (11) \quad \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} A_0 e^{\delta_0 t} \\ A_1 e^{\delta_1 t} \\ A_2 e^{\delta_2 t} \end{bmatrix}$$

Here δ_0 , δ_1 , and δ_2 are respectively characteristic roots for the symmetrical components. Inverse transformation of eq. (11) is as follows;

$$(12) \quad \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = 3A^{-1} \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = A^* \begin{bmatrix} i_0 \\ i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} A_0 e^{\delta_0 t} & A_1 e^{\delta_1 t} & A_2 e^{\delta_2 t} \\ A_0 e^{\delta_0 t} & A_1 e^{\delta_1 t - j2\pi/3} & A_2 e^{\delta_2 t + j2\pi/3} \\ A_0 e^{\delta_0 t} & A_1 e^{\delta_1 t + j2\pi/3} & A_2 e^{\delta_2 t - j2\pi/3} \end{bmatrix}$$

This equation shows three components i_0 , i_1 and i_2 are symmetrically distributed among three phases.

5. Circuit equations of three phase induction motors. As an example of the spiral vector symmetrical-component method, transient phenomena of the induction motor will be analyzed. Starting equations of the three-phase induction motor, whose model for the following analysis is shown in Fig. 2, are as follows;

For the stator

$$(13) \quad \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = (R_1 + l_1 p) \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + p \begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{bmatrix},$$

For the rotor, whose terminals are short-circuited.

$$(14) \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (R_2 + l_2 p) \begin{bmatrix} i_r \\ i_s \\ i_t \end{bmatrix} + p \begin{bmatrix} \lambda_r \\ \lambda_s \\ \lambda_t \end{bmatrix}$$

λ 's are flux linkages of the gap field. Multiplying each term with matrix $A/3$ from the left side, eqs. (13) and (14) become eqs. (15) and (16) respectively

$$(15) \quad \begin{bmatrix} v_{10} \\ v_{11} \\ v_{12} \end{bmatrix} = (R_1 + l_1 p) \begin{bmatrix} i_{10} \\ i_{11} \\ i_{12} \end{bmatrix} + p \begin{bmatrix} \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \end{bmatrix}, \quad (16) \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (R_2 + l_2 p) \begin{bmatrix} i_{20} \\ i_{21} \\ i_{22} \end{bmatrix} + p \begin{bmatrix} \lambda_{20} \\ \lambda_{21} \\ \lambda_{22} \end{bmatrix}$$

Flux linkages λ 's in eqs. (13)–(16) will be now expressed by currents and inductances. When center lines of phase a and phase r coincide, angle $\theta=0$ and the mutual inductance between them becomes its maximum value M . When $\theta \neq 0$, the mutual inductance is $M \cos \theta$. Flux linkage of primary three-phase windings produced by the revolving field in the air gap is thus given by

$$(17) \quad \begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{bmatrix} = M \begin{bmatrix} 1 & \cos(2\pi/3) & \cos(-2\pi/3) \\ \cos(-2\pi/3) & 1 & \cos(2\pi/3) \\ \cos(2\pi/3) & \cos(-2\pi/3) & 1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \\ + M \begin{bmatrix} \cos \theta & \cos(\theta+2\pi/3) & \cos(\theta-2\pi/3) \\ \cos(\theta-2\pi/3) & \cos \theta & \cos(\theta+2\pi/3) \\ \cos(\theta+2\pi/3) & \cos(\theta-2\pi/3) & \cos \theta \end{bmatrix} \begin{bmatrix} i_r \\ i_s \\ i_t \end{bmatrix}$$

Transforming these flux linkages into symmetrical components, we get

$$(18) \quad \begin{bmatrix} \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \end{bmatrix} = \frac{3}{2} M \begin{bmatrix} 0 \\ i_{11} + e^{j\theta} i_{21} \\ i_{12} + e^{-j\theta} i_{22} \end{bmatrix}$$

Main flux linkages of the secondary windings are similarly derived as follows;

$$(19) \quad \begin{bmatrix} \lambda_r \\ \lambda_s \\ \lambda_t \end{bmatrix} = M \begin{bmatrix} 1 & \cos(2\pi/3) & \cos(-2\pi/3) \\ \cos(-2\pi/3) & 1 & \cos(2\pi/3) \\ \cos(2\pi/3) & \cos(-2\pi/3) & 1 \end{bmatrix} \begin{bmatrix} i_r \\ i_s \\ i_t \end{bmatrix} \\ + M \begin{bmatrix} \cos(-\theta) & \cos(-\theta+2\pi/3) & \cos(-\theta-2\pi/3) \\ \cos(-\theta-2\pi/3) & \cos(-\theta) & \cos(-\theta+2\pi/3) \\ \cos(-\theta+2\pi/3) & \cos(-\theta-2\pi/3) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

Transforming these flux linkages into their symmetrical components, we get

$$(20) \quad \begin{bmatrix} \lambda_{20} \\ \lambda_{21} \\ \lambda_{22} \end{bmatrix} = \frac{3}{2} M \begin{bmatrix} 0 \\ i_{21} + e^{-j\theta} i_{11} \\ i_{22} + e^{j\theta} i_{12} \end{bmatrix}$$

Equations (18) and (20) give the following conclusions; Zero-sequence components currents do not produce main flux linkages. Symmetrical components are segregated from each

Fig. 3. Equivalent circuit of the induction motor for positive-sequence components.

Fig. 4. Equivalent circuit of the induction motor for negative-sequence components.

other in main flux linkage production, that is they are independent.

Inserting eqs. (18) and (20) into eqs. (15) and (16) respectively, we get the following circuit equations;

For zero-sequence components

$$(21) \quad \begin{cases} v_{10} = (R_1 + l_1 p) i_{10} \\ 0 = (R_2 + l_2 p) i_{20} \end{cases}$$

For positive-sequence components

$$(22) \quad \begin{cases} v_{11} = (R_1 + l_1 p) i_{11} + (3M/2)p(i_{11} + e^{j\theta} i_{21}) \\ 0 = (R_2 + l_2 p) i_{21} + (3M/2)p(i_{21} + e^{-j\theta} i_{11}) \end{cases}$$

For negative-sequence components

$$(23) \quad \begin{cases} v_{12} = (R_1 + l_1 p) i_{12} + (3M/2)p(i_{12} + e^{-j\theta} i_{22}) \\ 0 = (R_2 + l_2 p) i_{22} + (3M/2)p(i_{22} + e^{j\theta} i_{12}) \end{cases}$$

It should be noticed that in eqs. (21)–(23) each symmetrical component is separated or segregated from others. Thus these equations are normalized. Making the following replacement

$$(24) \quad i'_{21} = e^{j\theta} i_{21} = e^{j\omega_m t} i_{21}$$

, the first equation of eq. (22) becomes

$$(25) \quad v_{11} = R_1 i_{11} + (l_1 + 3M/2)p i_{11} + (3M/2)p i'_{21}$$

The second equation of eq. (22) is modified as below.

$$(26) \quad 0 = R_2 i_{21} + (l_2 + 3M/2)p i_{21} + (3M/2)(e^{-j\theta} p i_{11} - j\omega_m e^{-j\theta} i_{11})$$

Multiplying each term of eq. (26) with $e^{j\theta}$, we get

$$(27) \quad 0 = R_2 e^{j\theta} i_{21} + (l_2 + 3M/2)e^{j\theta} p i_{21} + (3M/2)(p - j\omega_m) i_{11}$$

Differentiating eq. (24), we get

$$(28) \quad p i'_{21} = j\omega_m i'_{21} + e^{j\theta} p i_{21}$$

Inserting eq. (28), eq. (27) becomes

$$(29) \quad 0 = R_2 i'_{21} + (l_2 + 3M/2)(p - j\omega_m) i'_{21} + (3M/2)(p - j\omega_m) i_{11}$$

Combining eqs. (25) and (29), the following matrix equation is obtained.

$$(30) \quad \begin{bmatrix} v_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + (l_1 + 3M/2)p & (3M/2)p \\ (3M/2)(p - j\omega_m) & R_2 + (l_2 + 3M/2)(p - j\omega_m) \end{bmatrix} \begin{bmatrix} i_{11} \\ i'_{21} \end{bmatrix}$$

This is the performance equation for positive-sequence components. Fig. 3 shows its equivalent circuit. Speed voltage e_{s1} in the figure is

$$(31) \quad e_{s1} = -j\omega_m[(3M/2)i_{11} + (l_2 + 3M/2)i'_{21}]$$

Making the following variable replacement

$$(32) \quad i'_{22} = e^{-j\theta}i_{22} = e^{-j\omega_m t}i_{22}$$

circuit equations of negative-sequence components in eq. (23) is similarly changed into the following equation.

$$(33) \quad \begin{bmatrix} v_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + (l_1 + 3M/2) & (3M/2)p \\ (3M/2)(p + j\omega_m) & R_2 + (l_2 + 3M/2)(p + j\omega_m) \end{bmatrix} \begin{bmatrix} i_{12} \\ i'_{22} \end{bmatrix}$$

This is the performance equation for negative-sequence components of the induction motor. The corresponding equivalent circuit is given in Fig. 4.

Speed voltage e_{s2} in the figure is

$$(34) \quad e_{s2} = j\omega_m[(3M/2)i_{12} + (l_2 + 3M/2)i'_{22}]$$

Characteristic equations of performance equations (30) and (33) obtained above are of the second order with respect to $p=d/dt$. And these equations are easy to solve. They are valid for both steady and transient states. Thus applicability of the symmetrical component method has been extended to analysis of asymmetrical operation of the three-phase induction motor. This is an example of the method, but the similar treatment can be also applied to other three-phase machines and circuits.

6. Symmetrical operation. When the operation under consideration is symmetric or balanced, zero-sequence and negative-sequence components are zero and only positive-sequence remains in eq. (12). Thus we have

$$(35) \quad i_a = i_1, \quad i_b = \alpha i_a, \quad i_c = \alpha^2 i_a$$

In this case the circuit equation for positive-sequence components in eq. (30) becomes identical to the circuit equation derived in terms of state variables of phase a expressed in spiral vectors. This is the phase segregation method.¹⁾ This circuit equation was used to analyze transient phenomena of the induction motor and the analytical results obtained were used to derive very superior control of induction motor torque.

References

- 1) S. Yamamura: Spiral vector theory of AC circuit and machine. Proc. Japan Acad., 65B, 142-145 (1989).
- 2) S. Bekku: On short-circuit currents of AC generator. J. of IEE of Japan, p. 805 (1927).