7. **Spiral Vector Theory of Salient-Pole Synchronous Machine**

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**Abstract:** Almost all electric energy of electric industries is generated by synchronous generators, which are either salient-pole synchronous generators or have saliency to some extent. As the rotor rotates, inductances of the three phase winding vary due to the saliency and it makes analysis of generator performance difficult. Even adequate computer simulations have not been provided by the conventional theories, such as the d,q axis theory. In this paper the spiral vector method will be applied to analyses of the salient-pole synchronous generator, and adequate analytical solutions and computer simulations will be derived.

**Key words:** Spiral vector; computer simulation of synchronous machine; analysis of salient-pole synchronous machine.

1. **Introduction.** The spiral vector method was proposed by the author (1)-(6). It can unify AC circuit theories of steady state and transient state. It is most suitable to analyses of AC machine transients. Analytical results obtained have led to superior control performances of AC motors. (1)-4) The paper will now apply the method to analyses of the salient-pole synchronous machine, whose analyses have been exclusively made by Park's equations based on the d,q axis theory, but they have neither provided adequate analytical solutions nor computer simulations of the synchronous machine. In this paper it will be shown that the spiral vector method provides both of them.

2. **Steady state analysis of salient-pole synchronous machine.** Fig. 1 shows the model of the salient-pole synchronous machine for the analysis which follows. It is the permanent-magnet-excited type or the field winding type. Self-inductances of the armature windings are functions of spacial angle $\theta$, which is the angle between the center line of winding of phase a and the direct axis of the salient-pole. Then we have

\[ \theta = \omega t + \varphi_d \] [elect. radians]

where $\omega$ is synchronous speed of the rotor. Then self-inductances of phases a, b and c are given by

\[ L_a = L + L' \cos (2\theta) = L + L' \cos (2\omega t + 2\varphi_d) \]
\[ L_b = L + L' \cos (2\theta - 4\pi/3) = L + L' \cos (2\omega t + 2\varphi_d - 4\pi/3) \]
\[ L_c = L + L' \cos (2\theta + 4\pi/3) = L + L' \cos (2\omega t + 2\varphi_d + 4\pi/3) \]

These inductances contain double frequency components. Mutual inductances between two phases among three phases a, b and c are similarly given as below.

\[ M_{ab} = M_{ab}' \cos (2\theta + \psi) \]
\[ M_{bc} = M_{bc}' \cos (2\theta + \psi - 4\pi/3) \]
\[ M_{ca} = M_{ca}' \cos (2\theta + \psi + 4\pi/3) \]
Angle $\psi$ is phase difference between double frequency variations of self-inductances of eq. (2) and mutual inductances of eq. (3) and is to be determined shortly in this paper. Main flux linkage $\lambda_{ga}$ of phase a is

$$\lambda_{ga} = -L_a i_a - M_{ab} i_b - M_{ca} i_c + \lambda \cos \theta$$

The last term of this equation is flux linkage from the magnetic poles of the rotor, which are the permanent magnet or the field winding with constant DC current. For symmetrical or balanced steady state operation three-phase armature currents are given by

$$i_a = \sqrt{2} |I_a| \cos (\omega t + \varphi_a)$$
$$i_b = \sqrt{2} |I_a| \cos (\omega t + \varphi_a - 2\pi/3)$$
$$i_c = \sqrt{2} |I_a| \cos (\omega t + \varphi_a + 2\pi/3)$$

Here $I_a$ is circular vector of current of phase a. Inserting eq. (5), eq. (4) becomes

$$\lambda_{ga} = -(L - M_{ab}) \sqrt{2} |I_a| \cos (\omega t + \varphi_a) + (\sqrt{2}/2) |I_a| [L' \cos (\omega t + 2\varphi_a - \omega_1) + M' \cos (\omega t + 2\varphi_a + \psi - \varphi_1 + 2\pi/3) + (\sqrt{2}/2) |I_a| [L' \cos (3\omega t + 2\varphi_a + \varphi_1) + M' \cos (3\omega t + 2\varphi_a + \varphi_1 + \psi - 2\pi/3) + M' \cos (3\omega t + 2\varphi_a + \varphi_1 + \psi)]] + \lambda \cos (\omega t + \varphi_a)$$

Triple frequency terms are given in the bracketed term second from the end, and in order for the triple-frequency terms to disappear the following conditions are sufficient.

$$L' = M', \quad \psi = -2\pi/3$$

Salient-pole synchronous machines are constructed so that these conditions are satisfied. From symmetry of the machine structure we have

$$M_{ab} = L \cos (2\pi/3) = -L/2$$

Inserting eqs. (7) and (8), $\lambda_{ga}$ of eq. (6) now becomes

$$\lambda_{ga} = -(3L/2) \sqrt{2} |I_a| \cos (\omega t + \varphi_a) - (3L'/2) \sqrt{2} |I_a| \cos (\omega t + 2\varphi_a - \varphi_1) + \lambda \cos (\omega t + \varphi_a)$$
Now variables are expressed in circular vectors, and then eq. (9) becomes

\[ \lambda_a = -\left(3L/2\right) \sqrt{2} |I_a| e^{(\omega t + \varphi_1)} - \left(3L'/2\right) \sqrt{2} |I_a| e^{(\omega t + \varphi_1 + 2\varphi_2 - 2\varphi_1)} + \lambda e^{(\omega t + \varphi_2)} \]

\[ = -\left(3L/2\right)i_a - \left(3L'/2\right)i_a e^{(\varphi_2 - \varphi_1)} - \lambda e^{(\omega t + \varphi_2)} \]

Here \( i_a = \sqrt{2} \hat{I}_a = \sqrt{2} |\hat{I}_a| e^{(\omega t + \varphi)} \) is circular vector of current of phase a. Circuit equation of phase a of the synchronous generator is now given by

\[ v_a = -R_i i_a - i_p i_a + \rho \lambda_a \]

\[ = -R_1 i_a - (l + 3L/2) p i_a - \left(3L'/2\right) e^{(\varphi_2 - \varphi_1)} p i_a + e_a \]

Here \( l_1 \) is primary leakage inductance and \( e_a \) is internal induced voltage given below.

\[ e_a = j \omega \lambda e^{(\omega t + \varphi)} = \omega \lambda e^{(\omega t + \varphi)}, \quad \varphi = \varphi_1 + \frac{\pi}{2} \]

Equation (11) contains phase a only, being segregated from other phases. In order for phase a to represent primary three phases, subscript a is changed to 1 and eq. (11) becomes

\[ v_1 = -R_1 i_1 - (l + 3L/2) p i_1 - \left(3L'/2\right) e^{(\varphi_2 - \varphi_1)} p i_1 + e_1 \]

In this equation inductance becomes eq. (14), which is a complex inductance.

\[ l_1 + 3L/2 - \left(3L'/2\right) e^{(\varphi - \varphi_1)} = L_1 - \left(3L'/2\right) e^{(\varphi - \varphi_1)}, \quad L_1 = l_1 + 3L/2 \]

Under steady states operator \( p \) becomes \( j \omega \) and variables are circular vectors, which are now expressed by capital letters with \( \cdot \) on top. Then eq. (13) becomes

\[ \dot{V}_1 = -R_1 \dot{I}_1 - j \omega (L_1 - \left(3L'/2\right) \cos (2\varphi - 2\varphi_1)) \dot{I}_1 + E_1 \]

\[ = -[R_1 + j \omega (3M'/2) \sin (2\varphi - 2\varphi_1)] \dot{I}_1 - j \omega (L_1 + \left(3L'/2\right) \cos (2\varphi - 2\varphi_1)) \dot{I}_1 + E_1 \]

Vector diagram corresponding to eq. (15) at \( t=0 \), is shown in Fig. 2. Resistance is increased by \( \omega (3M'/2) \times \sin(2\varphi - 2\varphi_1) \) due to saliency. And motor torque is also increased due to saliency by

\[ 3(P/2)(3L'/2) \sin (2\varphi - 2\varphi_1) |\hat{I}|^2 \quad [N\cdot m] \]

where \( P \) is number of poles. This is saliency torque or reluctance torque.

![Fig. 2. Circular vector diagram at \( t = 0 \) of the salient-pole synchronous generator.](image-url)
3. Transient state analysis of salient-pole synchronous machine. Under transient state there are plural number of characteristic roots, which are generally complex. Under symmetrical (or balanced) operation state variables of three phases are symmetrical. Under asymmetrical operation asymmetrical state variables are transformed by the spiral-vector symmetrical-component method into symmetrical spiral vectors. Symmetrical three phase currents under transient state are assumed to be expressed by the following equations.

\[ i_a = \sqrt{2} \left| I_a \right| e^{-it} \cos (\omega t + \varphi) \]
\[ i_b = \sqrt{2} \left| I_a \right| e^{-it} \cos (\omega t + \varphi - 2\pi/3) \]
\[ i_c = \sqrt{2} \left| I_a \right| e^{-it} \cos (\omega t + \varphi + 2\pi/3) \]

Inserting eq. (17), main flux linkage of phase a of eq. (4) becomes

\[ \lambda_{ga} = -(L-M_a)e^{-it} \sqrt{2} \left| I_a \right| \cos (\omega t + \varphi_a) - \left( \sqrt{2}/2 \right) I_a e^{-it} \left[ \cos (2\omega t - \omega' t + 2\varphi_a - \varphi_{1,2}) - M' \cos (2\omega t - \omega' t + 2\varphi_a + \varphi) \right] - \lambda \cos (\omega t + \varphi) \]

In order for \(2\omega + \omega'\) frequency terms to disappear, the condition of eq. (7) is sufficient. And the \(\lambda_{ga}\) of eq. (18) becomes

\[ \lambda_{ga} = -(3L/2)e^{-it} \sqrt{2} \left| I_a \right| \cos (\omega t + \varphi_a) - \left(3L'/2\right)e^{-it} \left[ \cos (2\omega t - \omega' t + 2\varphi_a - \varphi_{1,2}) - M' \cos (2\omega t - \omega' t + 2\varphi_a + \varphi) \right] + \lambda \cos (\omega t + \varphi) \]

Expressing this equation in spiral vectors, it becomes

\[ \lambda_{ga} = -(3L/2)e^{-it} \sqrt{2} \left| I_a \right| e^{j(\omega t + \varphi_a)} - \left(3L'/2\right)e^{-it} e^{j(2\omega t + \varphi_a)} - M' e^{j(2\omega t + \varphi_a)} + \lambda e^{j\theta} \]

Here \(i_a = \sqrt{2} \left| I_a \right| e^{-it} \) is spiral vector of current of phase a and \(i_a^*\) is its conjugate.

Now circuit equation of phase a is

\[ v_a = -R_i i_a - l_i p i_a + p\lambda g_a \]
\[ = -R_i i_a - (l_i + 3L/2) p i_a - (3L'/2) p (e^{j\omega t} i_a^*) + p(\lambda e^{j\theta}) \]

This equation contains only variables of phase a, which is segregated from other phases. Changing subscript a to 1, eq. (23) becomes

\[ v_1 = -R_i i_1 - (l_i + 3L/2) p i_1 - (3L'/2) p (e^{j\omega t} i_1^*) + p(\lambda e^{j\theta}) \]

This is valid both for steady and transient states.

When the generator is permanent-magnet-excited or its field current is kept constant, flux \(\lambda\) is constant. Then eq. (22) becomes

\[ e_1 = v_1 = -R_i i_1 - (l_i + 3L2) p i_1 + (3L'/2) p (e^{j\omega t} i_1^*) \]

where \(e_1\) is internal induced voltage of eq. (12) with 1 replacing a.
Let us write the general solution of eq. (23) as follows;

\[ i_t = i_{st} + \sqrt{2} |I_1| e^{i(\omega t + \psi)} \]

The first term is the transient state term and the second term is the steady state term, which was obtained in the previous section. Inserting eq. (24) into eq. (23), we get

\[ 0 = R_i i_t + (l_i + 3L/2) p i_t + (3L'/2)p(e^{j(\omega t + \psi)} i_t^*) \]

The last term is a time function, which behaves as a forcing function. As a first approximation, let us assume \( i_{st} = A e^{-\lambda t} \) for the forcing function. Then eq. (25) becomes

\[ R_i i_t + (l_i + 3L/2) p i_t = -(3L'/2)p[A e^{(-j\omega t + j\psi)}] \]

The forcing function is a spiral vector and general solution of this equation is given, as follows.

\[ i_{st} = A e^{-\lambda t} - \frac{(3L'/2)(-\lambda + j2\omega)A}{R_i + (-\lambda + j2\omega)(l_i + 3L/2)} e^{(-j\omega t + j\psi)} + \frac{\sqrt{2} |I_1| e^{j(\omega t + \psi)}}{R_1 + (-\lambda + j2\omega)(l_i + 3L/2)} \]

Here

\[ \lambda = \frac{R_i}{l_i + 3L/2} \]

and \( A \) is an arbitrary constant, which is to be determined by an initial condition. We can repeat the process by inserting the solution of eq. (27) into the forcing function of eq. (26). And then we get the same general solution with \( \lambda \) slightly modified, which contains an arbitrary constant \( A \). And so eq. (27) is a transient solution of good approximation. General solution of eq. (24) now becomes

\[ i_t = A e^{-\lambda t} - \frac{(3L'/2)(-\lambda + j2\omega)A}{R_i + (-\lambda + j2\omega)(l_i + 3L/2)} e^{(-j\omega t + j\psi)} + \frac{\sqrt{2} |I_1| e^{j(\omega t + \psi)}}{R_1 + (-\lambda + j2\omega)(l_i + 3L/2)} \]

The first and second terms are transient terms and contain \( A \). If at \( t=0 \) the following condition is satisfied

\[ i_t = f_0 i_I e^{j\psi} \]

\( A \) becomes zero and no transient occurs. Eq. (30) is the transientless condition of synchronous machine control.

4. **Transient state analysis of salient-pole synchronous machine with field winding.** When the synchronous machine is of the field winding type and transient current flows in the winding, transient state analysis becomes a little more complicated. Analysis in section 3, which was made about the machine with constant flux linkage of the field winding, is valid up to eq. (20) for analysis in this section.

When transient current flows in the field winding, the last term of eq. (20) becomes

\[ \lambda = M_f i_f e^{j\psi} \]

Thus eq. (22) becomes

\[ v_1 = -(R_i i_t - (l_i + 3L/2)p i_t - (3L'/2)p(e^{j(\omega t + \psi)}) + M_f p(i_f e^{j\psi}) \]
The circuit equation of the field winding is in a similar way derived, as follows,

\[ v_f = R_i f + (l_i + M_f) p i_f - (3 M_f / 2) p (i^*_f e^{i\theta}) \]

Eqs. (32) and (33) are circuit equations to be solved simultaneously.

5. Computer simulation of the synchronous generator. In the spiral vector method performance equations are written in terms of original state variables, without any variable transformation. Thus the equations are very suitable to make computer simulation of the synchronous machine, which can be incorporated with ease into simulation of the power system. In the conventional theories, such as the d, q axis theory or Park's equations, it seems that adequate computer simulation is not available.

Computer simulation of the synchronous generator will be given, as an example, of equations (32) and (33) in section 4. They are modified as below

\[ i_f = - \frac{v_f}{L_x p} - \frac{R_i}{L_x} i_f - \frac{3}{2} \frac{L'}{L_x} (e^{it^*_f} + \frac{M_f}{L_x} i_f e^{i\theta}) \]

\[ i_f = \frac{v_f}{(l_f + M_f)p} - \frac{R_f}{l_f + M_f} i_f + \frac{3}{2} \frac{M_f}{l_f + M_f} (e^{it^*_f} + \frac{M_f}{l_f + M_f} i_f e^{i\theta}) \]

![Fig. 3. Block diagram of computer simulation of salient-pole synchronous generator.](image)

The corresponding block diagram of these equations is given in Fig. 3.

6. Conclusion. The spiral vector method has been applied to analyses of the salient-pole synchronous machine. Both steady and transient states have been analyzed. It is shown that analytical power of the method is much better than the d, q axis method. It has provided computer simulations of the synchronous machine, which was missing in conventional theories based on the d, q axis method.

References
3) ——: ibid., ICEMD, Rumania (1986).