45. Spiral Vector Theory of Electric Circuit

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Abstract: Conventional electric circuit theories make use of two different expressions of state variables. Phasor or vector expression is used for steady state analysis and real value expression is used for transient state analysis. Due to the different expressions steady state and transient state solutions are not compatible and it causes inconvenience and awkwardness in analysis and computer simulation. The spiral vector method proposed in this paper, which makes use of exponential time function for expressing state variables, can treat electric circuit analysis in an unified way and opens up a new area of the unified circuit theory. The spiral vector method is a very powerful tool for analysis and control of AC machines also.¹⁾⁻⁴⁾

Key words: Spiral vector; spiral vector special solution; spiral vector theorem; spiral vector theory of AC circuit; phase segregation; FAM.

1. Introduction. The conventional electric circuit theories have been in use since the beginning of the century. They rely on two different expressions of state variables; phasor for steady state analysis and real value expression for transient state analysis. Thus steady state solutions and transient state solutions are not compatible and it is cumbersome and awkward to combine them in general solutions of circuit equations and in computer simulations. The author proposes the spiral vector method, which provides an unified way of expressing state variables, and unifies the conventional theories.

2. Definition of spiral vector. The spiral vector is defined as the following form of exponential time function.

$$i = Ae^{\delta t}, \quad \delta = -\lambda + j\omega \quad (1)$$

As time proceeds, i depicts a spiral in the complex plane of Fig. 1. When $\lambda = 0$, i depicts a circle in the complex plane and is called a circular vector. This represents steady state AC. When $\omega = 0$, it becomes $i = A \exp(-\lambda t)$. This represents attenuating DC. When $\lambda = \omega = 0$, i becomes $i = A$, which represents steady state DC. Thus the spiral vector can represent most kinds of state variables, which appear in electrical engineering.

3. Conversion between spiral vector and corresponding real value. SV of eq. (1) can be written as below.

$$i = |A| e^{j\varphi_A} e^{(-\lambda + j\omega)t} = |A| e^{-\lambda t} [\cos(\omega t + \varphi_A) + j \sin(\omega t + \varphi_A)] \quad (2)$$

A real value, which corresponds to i of eq. (2), can be chosen to be either of the following two equations.

$$i_{\text{real}} = |A| e^{-\lambda t} \cos(\omega t + \varphi_A) = \text{real value of } i \quad (3)$$

$$i_{\text{real}} = |A| e^{-\lambda t} \sin(\omega t + \varphi_A) = \text{real value of imaginary part of } i. \quad (4)$$

In SV method the first choice of eq. (3) is adopted. i_{real} of this choice is the projection of SV i on the real axis in the complex plane of Fig. 1. Thus we get the following rule or theorem.

SV Theorem 1. The real value expression of state variable is the real part of its SV

Fig. 1 (left). Spiral vector in complex plane.

Fig. 2 (right). Vector diagram of SVs in eq. (15) for $t = 0$.

expression; that is

$$i_{\text{real}} = \text{Re} [i] . \quad (5)$$

Since cosine is an even function, both $Ae^{(-\lambda+j\omega t)}$ and $Ae^{(-\lambda-j\omega t)}$ give the same i_{real} , according to this theorem. However, only SVs, which rotate counter-clockwise in the complex plane, shall be chosen. Thus we get the following rule or theorem.

SV Theorem 2. The spiral vector rotates counter-clockwise in the complex plane.

This theorem means that $\omega \geq 0$ in eq. (1). In the SV method SVs are expressed by lower case letters, while circular vectors are expressed by capital letters with $\cdot$ at their top. For example, SV of current is i and its circular vector is $\dot{I}$.

In the SV method we have $p\dot{I} = j\omega\dot{I}$ and $\dot{I}/p = \dot{I}/j\omega$, while in the phasor method we do not have such mathematical relations. The phasor method is nothing but a notation method. It is now replaced by the SV method. Huge amount of analytical results obtained by the phasor method are now available to the SV method, with slight change of symbols, such as $\dot{I} = Ie^{j\omega t}$, where I is phasor.

4. Example of SV method analysis. For an electric circuit consisting of capacitance, inductance and resistance in series the following circuit equation holds.

$$L \frac{d^2 i}{dt} + R \frac{di}{dt} + \frac{1}{C} i = \frac{d}{dt} v \quad (6)$$

Rewriting, this equation can be written as below.

$$ap^2 i + bpi + ci = pv \quad (7)$$

Let impressed terminal voltage be expressed by the following circular vector.

$$v = \sqrt{2} | \dot{V} | e^{j(\omega t + \varphi)} = \sqrt{2} V e^{j\omega t} = \sqrt{2} \dot{V} \quad (8)$$

Let us assume that the solution of eq. (7) is the following SV.

$$i = Ae^{\delta t} \quad (9)$$

For steady state δ becomes $j\omega$ and we have $pi = j\omega i$. And eq. (7) becomes

$$[(j\omega)^2 a + (j\omega) b + c] i = \sqrt{2} | \dot{V} | j\omega e^{j(\omega t + \varphi)} , \quad (10)$$

from which we get the following steady state solution i_s

$$i_s = \frac{\sqrt{2} | \dot{V} | j\omega}{-\omega^2 a + c + j\omega b} e^{j(\omega t + \varphi)} = \sqrt{2} I_s e^{j\omega t} = \sqrt{2} \dot{I}_s . \quad (11)$$

For transient solution, inserting eq. (9) into the homogenous equation of eq. (7), we get

$$a\delta^2 + b\delta + c = 0, \quad (12)$$

which is the characteristic equation. There are two characteristic roots, as follows.

$$\delta_1, \delta_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (13)$$

Corresponding to them the general transient solution is given below.

$$i_t = A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} \quad (14)$$

Here A_1 and A_2 are arbitrary constants, which are to be determined from two initial conditions. The general solution is given by

$$i = i_s + i_t = \frac{\sqrt{2} |\dot{V}| j\omega}{c - a\omega^2 + j\omega b} e^{j(\omega t + \varphi)} + A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t}. \quad (15)$$

It should be noticed here that all terms of the general solution are SVs.

By setting $t = 0$, SVs become stationary vectors in the complex plane. For example, for $t = 0$ the general solution of eq. (15) are written below.

$$i = \sqrt{2} I_s + A_1 + A_2 \quad \text{for } t = 0, \quad (16)$$

whose terms are depicted as stationary vectors in the complex plane of Fig. 2.

When $b^2 - 4ac > 0$, characteristic roots are different real roots and transient terms in eq. (15) are attenuating DCs.

When $b^2 - 4ac = 0$, δ_1 and δ_2 are equal real roots, and general transient solution becomes

$$i_t = A_1 e^{\delta_1 t} + A_2 t e^{\delta_1 t}. \quad (17)$$

When $b^2 - 4ac < 0$, the characteristic roots are complex roots, which are conjugate to each other. They can be written as

$$\delta_1, \delta_2 = -\lambda \pm j\omega'. \quad (18)$$

One of them gives a SV, which rotates clockwise in the complex plane and is against SV Theorem 2. It must be and can be discarded. By choosing $A_2 = 0$ we get the general transient solution as below.

$$i_s = A_1 e^{\delta_1 t} \quad (19)$$

A conjugate term does not contain any additional information and arbitrary constant A_1 of eq. (19) is complex and can accomodate two initial conditions.

We shall now derive electric power in terms of SVs. Electric power p is a scalar and is given below.

$$p = \text{Re}[v] \cdot \text{Re}[i] \quad (20)$$

General solution of current i of eq. (15) can be written as

$$i = \sqrt{2} |\dot{I}| e^{j(\omega t + \varphi_1)} + |A_1| e^{(-\lambda + j\omega')t + j\varphi'} \quad (21)$$

and we get

$$\text{Re}[i] = \sqrt{2} |\dot{I}| \cos(\omega t + \varphi_1) + |A_1| e^{-\lambda t} \cos(\omega' t + \varphi'). \quad (22)$$

Inserting eqs. (8) and (22), eq. (20) becomes

$$\begin{aligned}
p &= \sqrt{2} \mid \dot{V} \mid \cos(\omega t + \varphi) \cdot [\sqrt{2} \mid \dot{I} \mid \cos(\omega t + \varphi_1) + \mid A_1 \mid e^{-\lambda t} \cos(\omega' t + \varphi')] \\
&= \mid \dot{V} \mid \mid \dot{I} \mid [(\cos(\varphi - \varphi_1) + \cos(2\omega t + \varphi + \varphi_1))] + \frac{\mid \dot{V} \mid \mid A_1 \mid}{\sqrt{2}} e^{-\lambda t} \\
&\quad \times [\cos\{(\omega - \omega')t + \varphi - \varphi'\} + \cos\{(\omega + \omega')t + \varphi + \varphi'\}].
\end{aligned} \tag{23}$$

This is also given by

$$p = \frac{1}{2} Re [vi^* + vi]. \tag{24}$$

Here * indicates conjugate. Symmetrical three phase power is not given by $3p$. Phase angles φ , φ_1 , and φ' have phase angle difference of $2\pi/3$ among phases. And so powers of cumulative frequency are cancelled among three phases and becomes zero in three phase power. Thus symmetrical three phase power becomes

$$p_3 = \frac{3}{2} Re[vi^*]. \tag{25}$$

Constant power of eq. (25) is

$$P_3 = 3 \mid V \mid \mid I \mid \cos(\varphi - \varphi_1). \tag{26}$$

If Re sign is removed from eq. (25), it becomes

$$P_3 = 3 \mid V \mid \mid I \mid [\cos(\varphi - \varphi_1) + j \sin(\varphi - \varphi_1)]. \tag{27}$$

The first term is watt component of three phase power and the second term is wattless component of three phases power.

5. General spiral vector theory of electric circuit. From the results of chapter 4 general SV theory of electric circuit will be deduced in this chapter. The general circuit equation of the multi-mesh circuit with two terminals can be written as follows.

$$A(p) i = B(p) v \tag{28}$$

$A(p)$ and $B(p)$ are polynomials of p as follows.

$$A(p) = a_0 + a_1 p + a_2 p^2 + \dots + a_n p^n, \quad B(p) = b_0 + b_1 p + b_2 p^2 + \dots + b_m p^m. \tag{29}$$

Here a_0 , a_1 , $\dots$ and b_0 , b_1 , $\dots$ are in most cases fixed real values.

When v in eq. (28) is a circular vector of eq. (8), steady state current solution i_s is also a circular vector. Accordingly we have $p i_s = j\omega i_s$ and $p v = j\omega v$. Eq. (28) then becomes $A(j\omega) i_s = B(j\omega) v$

and we get

$$i_s = \frac{B(j\omega)}{A(j\omega)} v = \frac{B(j\omega)}{A(j\omega)} \sqrt{2} \mid \dot{V} \mid e^{j(\omega t + \varphi)} = \sqrt{2} \mid \dot{I}_s \mid. \tag{30}$$

i_s is a circular vector $\sqrt{2} \mid \dot{I}_s \mid$ and $A(j\omega)/B(j\omega)$ is an AC impedance.

Characteristic equation of eq. (28) is

$$A(p) = 0, \tag{31}$$

whose characteristic roots are denoted $\delta_1, \delta_2, \dots, \delta_n$. Then the general transient solution is

$$i_t = A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} + \dots + A_n e^{\delta_n t}. \tag{32}$$

Here $A_1, A_2, \dots$ are arbitrary constants, which are determined from n initial conditions. If δ_r is a r th-order multiple root, the following terms must be added to i_t of eq. (32).

$$i_t' = A_1' t e^{\delta_1 t} + A_2' t^2 e^{\delta_2 t} + \dots + A_{r-1}' t^{r-1} e^{\delta_{r-1} t} . \quad (35)$$

When coefficients of eq. (29) are real, complex roots become conjugate pairs. One of them rotates clockwise and must be discarded according to SV Theorem 2, by choosing its arbitrary coefficient equal to zero. Thus we get the following theorem.

SV Theorem 3. General circuit equation is given by

$$A(p) i = B(p) v . \quad (28)$$

Terminal voltage v is a circular vector of eq. (8). Then the general solution of current i is given by

$$i = \frac{B(j\omega)}{A(j\omega)} \sqrt{2} \mid \dot{V} \mid e^{j(\omega t + \varphi)} + A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} + \dots + A_n e^{\delta_n t} . \quad (34)$$

In this theorem $\delta_1, \delta_2, \dots, \delta_n$ are characteristic roots of $A(p) = 0$. Terms of the general solution in eq. (34), which rotate clockwise, must be discarded. The discard does not destroy generality of the solution, because of presence of conjugate roots, whose SVs rotate anti-clockwise. All terms of the general solution of eq. (34) are SVs. Both steady state and transient state solutions are written in the form of SVs, and thus Theorem 3 gives a unified way of treating both steady and transient states.

The unified theory is now extended to the case where terminal voltage is SV below.

$$v = \sqrt{2} \mid \dot{V} \mid e^{(-\lambda + j\omega t + j\varphi)} = \sqrt{2} \mid \dot{V} \mid e^{\delta t + j\varphi} \quad (35)$$

The general solution is given by the following theorem.

SV Theorem 4. General solution of eq. (28) is given by

$$i = \frac{B(\delta)}{A(\delta)} \sqrt{2} \mid \dot{V} \mid e^{\delta t + j\varphi} + A_1 e^{\delta_1 t} + A_2 e^{\delta_2 t} + \dots + A_n e^{\delta_n t} \quad (36)$$

The second term and terms thereafter are the general transient solution. The first term looks like a steady state solution, but is not a circular vector. Let it be called SV special solution. It will be now proved that $i = [B(\delta) / A(\delta)] v$ is a special solution. Inserting it into eq. (28), we get

$$A(p) \frac{B(\delta)}{A(\delta)} v = B(p) v . \quad (37)$$

Modifying this as follows;

$$\frac{A(p)}{A(\delta)} v = \frac{B(p)}{B(\delta)} v \rightarrow \frac{A(\delta)}{A(\delta)} v = \frac{B(\delta)}{B(\delta)} v \rightarrow v = v .$$

This holds. So it is proved that $i = [B(\delta) / A(\delta)] v$ is a solution.

Having obtained current solutions in SVs, electric power will now be expressed by them. Deducing from eqs. (24)–(27), which are electric power p for the example of circuit equation. (7), we can derive the following theorem.

SV Theorem 5. Instantaneous power p is given by

$$p = Re [v] \cdot Re [i] = \frac{1}{2} Re [v i^* + v i] . \quad (38)$$

Symmetrical three phase instantaneous power is given by

$$P_3 = \frac{3}{2} Re [v i^*] . \quad (39)$$

Fig. 3. L - R circuit. Fig. 4. Vector diagram of SV special solution i_s .

Now an example of SV Theorem 4 will be given. In the circuit of Fig. 3 impressed terminal voltage is SV. Then the circuit equation is

$$Ri + Lpi = \sqrt{2} \mid \dot{V} \mid e^{(-\lambda+j\omega)t+j\varphi}. \quad (42)$$

General solution for this is given by SV Theorem 4, as follows;

$$i = \frac{\sqrt{2} \mid \dot{V} \mid e^{(-\lambda+j\omega)t+j\varphi}}{R + (+\lambda + j\omega t)} + Ae^{-(R/L)t} = i_s + i_t. \quad (43)$$

The first term is a special solution i_s , which satisfies the following equation.

$$Ri_s + (-\lambda+j\omega)Li_s = \sqrt{2} \mid \dot{V} \mid e^{(-\lambda+j\omega)t+j\varphi}. \quad (44)$$

At $t = 0$ this equation gives the vector diagram in Fig. 4. Effective resistance is $R - \lambda L$ and inductance remains the same as L . When $R = \lambda L$, effective resistance becomes zero, and it looks like a kind of resonance.

6. Variable conversion and circuit equation. When expressions of state variables are changed, generally circuit equations or performance equations are also changed. However, the variable conversion between real value expression and SV expression does not change circuit equations concerned. The circuit equation of eq. (28) remains unchanged. This will be now explained.

From SV Theorem 1 we get

$$\frac{d i_{\text{real}}}{d t} = \frac{d \text{real}[i]}{d t} = \text{Re} \left[\frac{d i}{d t} \right] \quad (45)$$

and further more we get

$$p^n i_{\text{real}} = p^n \text{Re} [i] = \text{Re} [p^n i]. \quad (46)$$

From eq. (28) we get

$$\text{Re} [A(p)i] = \text{Re} [B(p)v] \quad (47)$$

and so we get

$$A(p)i_{\text{real}} = B(p)v_{\text{real}}. \quad (48)$$

Thus it is now proved that circuit equation (28) holds for both real value expression and SV expression of state variables.

It should be pointed out here that solutions of circuit equations are much more easily obtainable in SV expression than in real value expression.

References

- 1) S. Yamamura: Spiral vector theory of AV circuit and machine. Proc. Japan Acad., **65B**, 142-145 (1989).
- 2) —: Spiral vector method and symmetrical component method. *ibid.*, **67B**, 1-6 (1991).
- 3) —: Theory of ac motor analysis and control. ICEM, vol. 1, Pisa, Italy (1989).
- 4) —: Spiral Vector Theory of AC Circuits and Machine (book). Oxford University Press, Oxford (1992).
- 5) —: Spiral vector theory of salient-pole synchronous machine. Proc. Japan Acad., **68B**, 30-35 (1992).