A Note on a Potential Problem.

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The solution of Laplace's equation in a space between two infinite parallel plates and a circular cylinder with its axis placed parallel to, and equally apart from, the two plates may sometimes be required. Such a problem can generally be stated in the form of an integral equation and can be solved, under circumstances, by a suitable choice of a set of orthogonal functions, as was done by Strutt, whose procedure we shall follow.

In Fig. 1, M₁ and M₂ are two circles of equal radius a, coming in contact with each other at a point O, which is the centre of a greater circle L of radius b. We shall search for such r, satisfying the conditions (i) \( \nabla^2 V = 0 \) between L and M₁; (ii) \( V = P \), a constant, on L; (iii) \( V = Q \), another constant, on M₁ and M₂.

Let the densities of surface charges \( u_1 \) on M₁ and M₂ and \( u_2 \) (on L), which are not yet known, be expanded into

\[
\begin{align*}
    u_1 &= A_0 + \sum_{n=1}^{\infty} A_n \cos n\varphi, \\
    u_2 &= B_0 + \sum_{n=1}^{\infty} B_n \cos 2n\varphi,
\end{align*}
\]

then the potential \( V \) at any point \( C \)

\[
V = \int_{M_1}^{M_2} \log d \ln \Psi + \int_{M_1}^{M_2} \log d \ln \varphi + \int_{M_1}^{M_2} \ln d \ln d \varphi
\]  

(1) I had really an occasion to come upon the problem a few years ago, when I visited the Furukawa copper refinery at Nikko.

(2) Riemann-Weber: Part. Diff.gleich. 7 Aufl. 1491.

satisfies the condition (i). Now, the expansions of \( \log d_1, \log d_2, \) and \( \log d_3 \) into the orthogonal functions \( \cos n \psi \) and \( \cos n \theta \) are

\[
\log d = \log b - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{r}{b} \right)^k \cos k (\psi - \theta),
\]

\[
\log d_1 = \log r_1 - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{r_1}{r_2} \right)^k \cos k (\theta_1 - \varphi_1),
\]

\[
\log d_2 = \log r_2 - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{r_1}{r_2} \right)^k \cos k (\varphi + \theta_2).
\]

From these the integrals in (2) can easily be calculated, and the conditions (ii) and (iii) give

\[
P_{2n} = 2\alpha \cdot \lambda_n \log b + B_n b \log b
\]

\[
0 = -\frac{B_n}{2} - 2\alpha \left( \frac{\alpha}{b} \right)^n \frac{1}{2l} \left\{ \lambda_n + \sum_{k=n}^{\infty} \frac{(2l - k)!}{(2l - n)!} \lambda_k \right\}
\]

\[
\frac{Q}{\pi} = 2\alpha B_n \log b - b \sum_{k=1}^{\infty} \left( \frac{\alpha}{b} \right)^k \frac{B_n}{2l} + 2\alpha \cdot \lambda_n \log (2\pi)
\]

\[
+ 2\alpha \cdot \lambda_n \log b - a \sum_{k=1}^{\infty} \frac{A_k}{k} \left( -\frac{1}{2} \right)^k
\]

\[
0 = -\frac{A_n}{n} - \sum_{k=1}^{\infty} \frac{\alpha}{b} \sum_{m=0}^{k} \binom{k}{m} \frac{B_n}{2l} \frac{(2l)!}{(2l - n - m)!} - 2\alpha \cdot \lambda_n \left( -\frac{1}{2} \right)^n
\]

\[
- a \sum_{k=1}^{\infty} \frac{A_k}{k} \left( -\frac{1}{2} \right)^{k+1} \frac{(n+k-1)!}{(k-1)! n!}
\]

which is an infinite set of simultaneous equations to determine the unknown coefficients \( A_k \)'s and \( B_n \)'s. For values of \( x \left( = \frac{\alpha}{b} \right) \) such that \( x \leq \frac{1}{4} \), first few coefficients will give a tolerable approximation. Computation shows

\[
A_1/A_0 = 0.77 + 5.9x^4, \quad A_2/A_0 = -0.25 + 6.8x^4
\]

\[
A_3/A_0 = 0.05 + 0.1x^4, \quad A_4/A_0 = 0.02 - 0.3x^4
\]
With these coefficients, the desired function $V$ is

$$V = 2\pi b B_0 \log b - \sum_{k=1}^{\infty} \frac{a_0}{2k} \left( \frac{a}{r} \right)^k B_{2k} \cos 2k\theta + 2\pi a A_0 \log (r/r_0)$$

$$- \sum_{k=1}^{\infty} \frac{a_1}{k} A_{2k} \left( \frac{a}{r} \right)^k \cos k\theta - \sum_{k=1}^{\infty} \frac{a_2}{k} A_{4k} \left( \frac{a}{r} \right)^k \cos 2k\theta.$$  

(6)

Now the problem of Fig. 1 can easily be transformed into that which was proposed at the top of this note. Conformal transformation by reciprocal radius $z = \frac{b^2}{r}$ brings the point $(\rho, \theta)$ to the point $(\frac{b^2}{\rho}, -\theta)$.

![Diagram](image)

and the two straight lines $z = \pm c$ to the two circles of equal radius $a = \frac{b^2}{2c}$ (Fig. 2). Therefore (6) applies to the problem in question as well. In fact, (6), with the substitution $a = \frac{b^2}{2c}$, represents the potential at a point $(\rho = \frac{b^2}{r}, \theta)$. As a numerical example, the values of $V$ for $c = 2b$, $P = 10$, $Q = 0$ were computed, and the equipotential curves for this case were added to the figure.