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1. In the mean-value theorem of the Differential Calculus

$$f(x+h) = f(x) + hf'(x+\theta h), \quad 0<\theta<1,$$

$\theta$ may be regarded as a function of $x$ and $h$.

In a paper, published in the Bulletin of the Calcutta Mathematical Society,\(^{(1)}\) I gave a method of expanding $\theta$ in a series of ascending powers of $h$ and calculated the coefficients up to the term containing $h^7$ and gave the equation for determining the coefficient of $h^n$. In the present paper, I discuss the nature of this coefficient of $h^n$ and give the expansion of $\theta$ in some special cases.

The suffixes 1, 2, 3, ..., in $f(x)$ are used to denote successive differentiation with respect to $x$.

2. Taking

$$\theta = A_0 + A_1 h + A_2 h^2 + \cdots + A_n h^n + \cdots,$$

we have

\[
A_0 = \frac{1}{2},
\]

\[
A_1 = \frac{1}{24} \frac{f_2}{f_4},
\]

\[
A_2 = \frac{1}{2} \left[ \frac{1}{24} \frac{f_4}{f_4} - \frac{1}{32} \frac{f_2 f_4}{f_4^2} + \frac{1}{192} \frac{f_2^3}{f_4^2} \right],
\]

\[
A_3 = \frac{1}{3} \left[ \frac{11}{20} \frac{f_6}{f_4} - \frac{3}{32} \frac{f_2 f_6}{f_4^2} + \frac{11}{192} \frac{f_2^3 f_4}{f_4^2} \right],
\]

\[
A_4 = \frac{1}{480} \left[ \frac{13}{16} \frac{f_8}{f_4} - \frac{43}{32} \frac{f_2 f_8}{f_4^2} + \frac{7}{16} \frac{f_2^2 f_4}{f_4^2} - \frac{1}{32} \frac{f_2^4}{f_4^2} \right],
\]

\[
A_5 = \frac{1}{5} \left[ \frac{19}{896} \frac{f_{10}}{f_4} - \frac{31}{384} \frac{f_2 f_{10}}{f_4^2} + \frac{15}{64} \frac{f_2^2 f_4 f_8}{f_4^2} - \frac{55}{108} \frac{f_2^3 f_4^2}{f_4^2} + \frac{5}{16} \frac{f_2 f_4^3}{f_4^2} - \frac{5}{384} \frac{f_2^4 f_4}{f_4^2} + \frac{185}{1152} \frac{f_2^5}{f_4^2} \right].
\]

3. From the values of the coefficients, given above, and also from the general equation, given in the previous paper, it is easy to see that $A_n$ (the coefficient of $h^n$) does not contain any derivative higher than the $(n+2)$th and lower than the second. The derivative of the 2nd order i.e. $f_4(x)$ (written as $f_2$) and its powers appear only in the denominators of the various terms and the denominators do not contain any derivative, other than the second. In each term of an expression for a coefficient, the derivatives occur in such a way that the suffix or the sum of the suffixes of the derivative or the derivatives in the numerator, diminished by the suffix or the sum of the suffixes of the derivative or the derivatives in the denominator must be equal to the suffix of the coefficient. As for instance, if we consider the expression for $A_6$, the first term contains $\frac{f_7}{f_2}$; it is clear that the suffix of the numerator, diminished by the suffix of the denominator, is 5 and that is the suffix of the coefficient $A_6$. In the second term, we have $\frac{f_3f_8}{f_2^2}$ and therefore the sum of the suffixes of the derivatives in the numerator, which is 9, diminished by the sum of the suffixes of the derivatives in the denominator i.e. 4, gives 5. This is true for all the terms and the expressions for all the coefficients obey this law. Also the power of the derivative $f_2$ in the denominator is equal to the number of derivative factors, occurring in the numerator. As for example, we have $\frac{f_3^2f_8}{f_2^4}$ in the 4th term of the expression for $A_6$; thus the number of factors in the numerator is 4 and that is equal to the power of $f_2$ in the denominator.

Thus we have

$$A_n = \frac{1}{n!} \left[ a_0 \frac{f_{n+2}}{f_2} + \beta_0 \frac{f_3f_{n+1}}{f_2^2} + \frac{f_4f_n}{f_2^2} + \beta_2 \frac{f_3f_{n+1}}{f_2^2} + \gamma_0 \frac{f_2^2f_n}{f_2^5} + \gamma_1 \frac{f_3^2f_{n-1}}{f_2^5} + \gamma_2 \frac{f_4^2f_{n-2}}{f_2^5} + \gamma_3 \frac{f_5^2f_{n-3}}{f_2^5} + \ldots \right]$$
+ \delta_0 \frac{f_3^n f_{n-1}}{f_z^4} + \delta_1 \frac{f_2^n f_{n-2}}{f_z^2} + \delta_2 \frac{f_1^n f_{n-3}}{f_z^1} + \ldots

+ \ldots \ldots \ldots

\ldots + \lambda_0 \frac{f_3^n f^3_z}{f_{n-1}} + \mu_0 \frac{f_3^n}{f_n^z}

\]

where \( \alpha_0, \beta_0, \beta_1, \beta_2, \beta_3, \ldots, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots, \delta_0, \delta_1, \delta_2, \ldots, \lambda_0, \mu_0 \) are all constants and functions of \( n \).

4. Following the method of the previous paper, we have

\[
\alpha_0 = \frac{2^{n+1} - (n+2)}{(n+1)(n+2)2^{n+1}},
\]

\[
\beta_0 = -\frac{3.2^{n+1} + (n+1)(n-6)}{3(n+1)2^{n+2}},
\]

\[
\beta_1 = \frac{3.2^{n+1} + n(n-4)}{3.2^{n+2}},
\]

\[
\beta_2 = -\frac{5n.2^n + n(n-1)(11n+42)}{3.5.2^{n+4}},
\]

\[
\beta_3 = -\frac{3.5n(n+1)2^{n+3} + n(n-1)(11n+2)(13n+54)}{3.5.7.2^{n+4}},
\]

\[
\beta_4 = -\frac{7n(n+1)(11n+2)(2n+3) + n(n-1)(11n+2)(13n+3)(19n+90)}{3.5.7.2^{n+5}}
\]

\[
\gamma_0 = \frac{1}{3.2^{n+1} - 3.2^n - \frac{1}{2} n(n^2 - 27n + 56)}
\]

5. When \( f(x) = e^x \), all the derivatives are equal, each being equal to \( e^x \) and therefore by the rule, "number of derivative factors in the numerator of each term must be equal to the power of \( f_z \) in the denominator," we see that \( \theta \) is a function of \( h \) only—-a result, which is otherwise evident.

In this case

\[
A_0 = \frac{1}{2},
\]

\[
A_1 = \frac{1}{24},
\]

\[
A_2 = 0.
\]
Thus
\[ A_3 = -\frac{1}{3} \cdot \frac{1}{480}, \]
\[ A_4 = 0, \]
\[ A_5 = \frac{1}{5} \cdot \frac{1}{1512}, \]
\[ A_6 = 0, \]
\[ \ldots \ldots \ldots \]

Thus
\[ \theta = \frac{1}{2} + \frac{1}{24} - \frac{1}{3} \cdot \frac{1}{480} \cdot h^3 + \frac{1}{5} \cdot \frac{1}{1512} \cdot h^5 - \ldots \ldots \]

6. When \( f(x) = \sin x \), we have
\[ A_0 = \frac{1}{2}, \]
\[ A_1 = \frac{1}{24} \cot x, \]
\[ A_2 = -\frac{1}{2} \cdot \frac{1}{24} \left[ 1 + \cot^2 x \right], \]
\[ A_3 = \frac{1}{3} \left[ \frac{19}{320} \cot x + \frac{11}{192} \cot^3 x \right], \]
\[ A_4 = -\frac{1}{4} \cdot \frac{17}{480} + \frac{31}{240} \cot^3 x + \frac{3}{32} \cot^5 x, \]
\[ A_5 = \frac{1}{5} \left[ \frac{309}{2688} \cot x + \frac{475}{1728} \cot^3 x + \frac{185}{1152} \cot^5 x \right], \]
\[ A_6 = -\frac{1}{6} \cdot \frac{29}{896} + \frac{689}{2688} \cot^2 x + \frac{513}{1152} \cot^4 x + \frac{85}{384} \cot^6 x \],
\[ \ldots \ldots \ldots \ldots \]

Putting \( x = \frac{\pi}{2} \), we get
\[ A_3 = \frac{1}{2}, \]
\[ A_4 = 0, \]
\[ A_5 = -\frac{1}{2} \cdot \frac{1}{24}, \]
\[ A_6 = 0, \]
Thus we have

\[
A_4 = -\frac{1}{480} \frac{17}{4}, \\
A_5 = 0, \\
A_6 = -\frac{1}{896} \frac{29}{6}.
\]

Thus we have

\[
\theta = \frac{1}{2} - \frac{1}{24} h^2 - \frac{1}{480} h^4 - \frac{1}{896} h^6 - \ldots
\]