On the Iteration of Algebraic Functions I.

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The theory of the iteration of rational functions is deeply developed by Fatou, Julia and others recently.\(^{(1)}\)

On the other hand we know little literatures on that of algebraic functions.\(^{(2)}\)

We shall give here and in the subsequent papers some properties of such an iteration.

Since algebraic functions are many valued, the behaviours of the iteration even in a vicinity of a fixed point is not the same with those of the iteration of a uniform function.

We shall first consider such behaviours about a fixed point.

Throughout this paper we consider the iteration from the point of view of studying some conditions for which the consequents and antecedents of the iteration form a properly discontinuous group on the Gaussian plane.

A necessary and sufficient condition for which the iteration may be a discontinuous group only in a vicinity of a fixed point, considering only a branch of the function under some restriction within it, is given in Part I.

In Part II we shall give some miscellaneous results on these iterations.

I

Let \( y = \Phi(x) \) be an algebraic function of \( x \) defined by

\[
P(x, y) = 0,
\]

where \( P(x, y) \) is a polynomial with respect to \( x \) and \( y \), and

\[
\Phi(x) = a_0 + a_1(x-a) + \ldots.
\]


\(^{(2)}\) Fatou has given in Bull. Soc. math. France (1925) some results on the iteration of algebraic functions by interesting examples.
be its \(v\)-branches at a point \(a\), where \(\lambda_i\) are all rational and positive.

We call such a point (\(\infty\) included) as \(\psi_p(x) = a + \alpha_n(x-a)^\lambda_0 + \ldots\)
for at least one of \(v\)-branches a fixed point, which we can consider as the origin by a simple transformation.

In a sufficiently small vicinity \(V\) of a fixed point 0 such that the function has no other branch points than the origin we have the expansion of the form

\[ y = \psi(x) = ax^\lambda + \ldots \]

for a branch \(\psi(x)\) of \(\Phi(x)\) which is zero at the origin.

We consider as the iteration of an algebraic function \(\Phi(x)\) only such iterated functions

\[ \Phi(x), \Phi(\Phi(x)) = \Phi_1(x), \ldots, \Phi(\Phi_{-1}(x)) = \Phi_n(x), \ldots \]

and

\[ \Phi^{-1}(x) = \Phi_1(x), \Phi^{-1}(\Phi^{-1}(x)) = \Phi_2(x), \ldots, \Phi^{-1}(\Phi_{-1}(\Phi_{-1}(x))) = \Phi_{n-1}(x), \ldots \]

each defined by \(P(x, y) = 0, P_1(x, y) = 0, \ldots P_n(x, y) = 0, \ldots\) and \(P_1(x, y) = 0, \ldots P_n(x, y) = 0, \ldots\) obtained by eliminating \(y_1, y_2, \ldots\) from \(P(x, y) = 0, \ldots P_n(x, y) = 0, \ldots\) and from \(P(y, x) = 0, P(y, y_1) = 0, \ldots P(y, y_n) = 0, \ldots\)
respectively.

We consider in the latter part such a kind of iterated functions \(\Phi^{-1}(\Phi(x)), \ldots, \Phi_{-1}(\Phi_n(x)), \ldots\) defined by eliminating \(y_1, y_2, \ldots\) from \(P(x, y_1) = 0, P(y, y_1) = 0, \ldots y_1 = \Phi_n(x), \ldots, y_1 = \Phi_n(y), \ldots\) respectively.

Now a fixed point of the iteration can be obtained by the equation

\[ P(x, x) = 0. \]

Some of the branch points or coincident points of the iterated functions \(\Phi_1(x)\) and of the inversely iterated functions \(\Phi_{-1}(x)\) appear among the antecedents and consequents respectively, up to the rank \(i-1\) included, of the branch points or coincident points \(c\) of \(\Phi_1(x)\) or \(\Phi_{-1}(x)\).

For the branch points and coincident points \(c\) of \(\Phi_1(x)\) or \(\Phi_{-1}(x)\) are the values \(c\) for which the equation \(P(x, y) = 0\) or \(P(y, x) = 0\) respectively has two equal roots. In the same way the branch points and coincident points of \(\Phi_1(x)\) or \(\Phi_{-1}(x)\) are the values \(c'\) for which the equation \(P_1(c', y) = 0\) or \(P_1(y, c') = 0\) respectively has two equal roots. This last equation is equivalent to the system of the equations

\[ P_{-1}(c', z) = 0, P(x, z) = 0 \text{ or } P_{-1}(c', c') = 0, P(x, z) = 0, \text{ respectively.} \]
It has a double root if one or the other of the two equations has a double root; if it is the second, $z$ coincides with a branch point or a coincident point $c$ of $\Phi(x)$ or $\Phi^{-1}(x)$, and we have $c' = \Phi_{-i-1}(c)$ or $c' = \Phi_{i-1}(c)$.

If it is the first equation which has a double root, $c'$ is a branch point or a coincident point of the function $\Phi_{i-1}(x)$ or $\Phi_{-i-1}(x)$.

It follows that if the proposition enunciated is true when we replace $i$ by $i-1$ it is also true for $i$, for the branch points or the coincident points of $\Phi_i(x)$ or $\Phi_{-i}(x)$ are the points $c_i, \Phi_{-i}(c), \ldots, \Phi_{-i-1}(c)$ or $c, \Phi_i(c), \ldots, \Phi_{i-1}(c)$ respectively.

For $i=1$ the proposition is evidently true, hence it is true in general.

If $\lambda \neq 1$ in (1) then we have two cases to be distinguished.

(I) $0 < \lambda < 1$. We consider in this case the inverse function of the branch $\varphi(x)$ within a vicinity $V$, of the origin, whose value is equal to zero at the origin.

We take a vicinity $V\prime$ of the origin for the inverse function so that it does not contain other branch points than the origin and it consists of a domain which is common to all the branches of the inverse function of the branch $\varphi(x)$ within $V\prime$, which can be obtained by the finitely-valuedness of the function.

Thus in $V\prime$, we have

$$z = \varphi^{-1}(y) = \beta_1 y^n + \ldots$$

and $\mu > 1$.

We can reduce this case (I) to the case (II) by considering $\varphi_{-i}(y)$ instead of $\varphi(x)$ within $V\prime$.

(II) $\lambda > 1$. We consider the value of the differential coefficient of $\varphi(x)$ at the origin.

First if the branch points of $\varphi(x)$, the $i$-th iterated function of the element $\varphi(x)$ in $V\prime$, accumulate in an infinite number to the origin, then an infinite number of these will perhaps be the antecedents of the branch points or coincident points of $\varphi(x)$.

Since $\Phi(x)$ is algebraic, these antecedents belong to the infinite number of iterated functions $\varphi_i(x)$, $i = i_1, i_2, \ldots$, and the antecedents of some point accumulate in an infinite number to the origin. But I have no strict proof for it now.

Throughout the following cases we exclude the case in which the branch points of $\varphi_i(x)$ or of $\varphi_{-i}(x)$ accumulate to the origin, and we denote this restriction by (II).
If the branch points of \( \varphi_i(x) \) do not accumulate to the origin, then there exists a sufficiently small vicinity \( V_\eta \) of the origin within which no branch point of \( \varphi_i(x) \) appears, and we have the expansion

\[
y_i = \varphi_i(x) = \alpha_i^{k} + \cdots + \alpha_i^{N-1} x^k + \cdots \quad \text{in } V_\eta
\]

for \( i = 1, 2, \ldots \).

Now in \( V_\eta \) contained in \( V_* \) the difference quotient

\[
\left| \frac{\varphi(x)}{x} \right| < \eta, \quad \eta < 1,
\]

for any values of the branches of \( \varphi(x) \) in \( V_* \).

Thus within \( V_\eta \) we have

\[
\left| \frac{\varphi_n(x)}{x} \right| < \eta^n
\]

for any values of the branches of \( \varphi(x) \), where \( \varphi_n(x) \) denotes the \( n \)-th iterated function of \( \varphi(x) \).

Thus there exists in this case an infinite number of consequents of a point sufficiently near to the origin in \( V_\eta \) accumulating to the origin.

(III) If \( \lambda = 1 \) we have to distinguish several cases.

Now let \( V_\eta \) be a vicinity of the origin in which all \( \varphi_i(x) \), \( i = 1, 2, \ldots \) have no branch point other than the origin, then we have the expansion

\[
y_i = \varphi_i(x) = \alpha_i x + \cdots
\]

in \( V_\eta \).

(III a) \( |\alpha_i| > 1 \). In this case we consider as before the inverse function of \( \varphi(x) \), that is

\[
x = \varphi_i^{-1}(y) = \beta_i y + \cdots ,
\]

where \( |\beta_i| < 1 \) within \( V_\eta \) in which \( \varphi_i^{-1}(y) \) has no other branch points than the origin. Further we assume that \( V_\eta \) does not contain any branch point for \( \varphi_i^{-1}(y) \) other than the origin.

Thus we can reduce to the case (III b) by considering \( \varphi_i^{-1}(y) \) instead of \( \varphi(x) \).

(III b) \( |\alpha_i| < 1 \). In this case we have in sufficiently small vicinity of the origin

\[
\left| \frac{\varphi(x)}{x} \right| < \eta, \quad \eta < 1,
\]

for any values of the branches of \( \varphi(x) \).

Thus we have again under the assumption that there exists no other branch points than the origin in \( V_\eta \).
Hence in all the cases (III a), (III b) above mentioned, an infinite number of consequents or antecedents of a point in some vicinity of the origin converges to the origin.

(III c) \(|\alpha_i|=1\) and \(\alpha_i = e^{i\theta}, \theta\) being an irrational number.

We consider the inverse function

\[ x = \varphi_-(y) = e^{-i\theta}y + \ldots \]

within \(V_\theta\), and assume that there exists a vicinity \(V_\theta\) in which \(y = \varphi_i(x)\) and \(y = \varphi_-(x), i=1, 2, \ldots\) have no branch points other than the origin in \(V_\theta\).

We will give here a lemma which will be used in this case.

Lemma. If \(y = \alpha x + \ldots, \alpha \neq 0\), is \(p\)-valued within a vicinity \(V_\theta\), then the inverse function

\[ x = \frac{1}{\alpha} y + \ldots \]

is also \(p\)-valued within some vicinity \(V_\theta\) under the condition that there exists no other branch point than the origin of the function \(y = \alpha x + \ldots\) and of its inverse function within it.

Proof. Let \(y = \alpha x + \beta x^2 + \ldots, \lambda > 1\) be \(p\)-valued in \(V_\theta\).

Then by putting \(Y = y^\lambda, X = x^\lambda\), we have

\[ Y^\lambda = X^\lambda [\alpha + \beta X^{\lambda - 1} + \ldots]. \]

In a small vicinity \(|X'| < \delta, \quad Y = e^{i\theta}X' [\alpha + bX'^{\rho} + \ldots] \quad j=1, 2, \ldots \mu. \]

Hence \(X = e^{-i\theta}Y' [c + dY'^{\mu} + \ldots] \quad \text{in } |Y'| < \delta. \)

Thus

\[ X^\lambda = Y^\lambda [c' + d'Y'^{\rho} + \ldots]. \]

Hence

\[ z = c'y + d'y^\gamma + \ldots \quad \eta > 1. \]

Therefore \(x\) is a \(p\)-valued function of \(y\) in \(V_\theta\).

Now consider the function (1), with \(x_1\) and \(x_0\) instead of \(y\) and \(x\) respectively,

\[ x_1 = e^{i\theta}x_0 + bx_0^\gamma + \ldots \quad \gamma > 1, \]

\(\gamma\) being rational numbers.
Assume that (1) is \( m \)-valued in a vicinity of the origin.

Since we can transform the two variables \( x_1, x_0 \) by the same function
\( x_1 = y(y_1), \quad x_0 = y(y_0), \)
multiplicity or not, into the two variables \( y_1, y_0 \) in
the process of iteration, we put
\[
    y_1 = y_1^m
    \quad x_0 = y_0^m,
\]
and we have
\[
y_1 = e^{\theta y_1^m} + b y_1^m + \ldots \quad m' > m.
\]

\( m' \) being integers.

Hence we have \( m \) functions about the origin
\[
y_j = e^{\frac{\theta_j}{m_j}} = y_j(1 + b_j y_j^m + \ldots)
\]
where \( j = 1, 2, \ldots, m \), and \( h_j(y_j) \) being regular and \( h_j(0) = 0 \), which are regular
in a vicinity \( V' \) of the origin.

Those \( m \) functions \( y_1, y_2, \ldots, y_m \) can be used for our iteration.

Similarly we have \( m^2 \) functions
\[
y_{j0} = e^{\frac{\theta_j}{m_j} + \frac{\theta_0}{m_0}} y_0(1 + h_0(y_0))
\]
when the iteration is performed twice, and so on.

Now we consider the case (IIIC).

Since \( x_1 \) is in general a many valued function, the iteration of
\( x_1 = e^{\theta x_1} + \ldots \)
is equivalent to that of the system of functions
\( y_j = e^{\theta_j y_j^m} + \ldots \) for every \( j = 1, 2, \ldots, m \).

Thus we obtain in general an indefinitely many different system
of iterated functions in the vicinity of the origin.

Consider any one system of the iteration
\[
y_{j0}, y_{j1}, \ldots, (y_{jk}(y_0)), \ldots,
\]
\( j_0, j_1, \ldots \) being some one of \( 1, 2, \ldots, m \).

To this system there corresponds the iteration
\[
y_1 = e^{\theta_1 y_1} + \ldots
y_2 = e^{\theta_2 y_2} + \ldots
\ldots
y_k = e^{\theta_k y_k} + \ldots
\ldots
\]
all \( \theta_0, \theta_1, \ldots, \theta_k \), \ldots being rational numbers different from each other.
Thus we have a family of functions

\[ u_n = t_n(y_0) = \frac{y_n}{e^{\alpha_n}} = y_0 + \ldots \]

which are mono-valued (schlicht) within some vicinity \( V_n \) of the origin.

Hence we can apply Julia's research on this family of functions.

All the functions \( u_n \) have the derivative 1 at the origin and any of them do not take its values in \( V_n \) more than once.

Hence we can apply Koebe's theorem and we obtain that when \( y_0 \) describes the curve \( V \), the boundary of \( V_n \), the point \( u_n \) describes a closed curve \( U_n \), whose shortest distance from the origin is larger than a fixed constant \( d > \frac{1}{4(\frac{1}{2} - 1)} \rho \), \( \rho \) being the radius of a circle contained in \( V_n \).

Similarly the upper limit of \( |u_n| \), which is independent of \( n \), can be assured from the theorem on the mono-valued functions.

Hence we can conclude that the origin is within a region \( U \) which is simply connected, and interior to all the common parts of \( U_n, n=1,2,\ldots \).

\( U \) can be transformed bi-univocally into itself by \( u_n = t_n(y_n) \) and by the branch of \( y_0 = e^{-\alpha_n}(u_n) \), which is zero at the origin.

Now \( U \) can be mapped conformally on the interior of a circle with the center at the origin by \( y_0 = \sigma(y_0) \).

The transformed relation between \( v_n \) and \( v_n = \gamma(v_0) \), where \( v_0 \) corresponds to \( v_n \) will be the rotation \( v_n = v_0 e^{\alpha_n} \), \( n=1,2,\ldots \) which, conserving all circles with the center at the origin, conserves an infinite number of analytic curves bounding and containing \( U \).

We can in the vicinity of the origin find a regular function \( \sigma = \sigma(y_0) \) of \( y_0 \) such that the relation between \( v_n \) and \( v_n \) satisfies

\[ v_n = e^{\alpha_n} v_0. \]

Thus \( \sigma(y_0) \), which is regular at the origin satisfies the functional equation of Schröder \( \sigma(t_n(y_0)) = e^{\alpha_n} \sigma(y_0) \).

Inversely if it has a solution which is regular at the origin, all the points \( y_n \) near to the origin has all its consequences on the analytic curve which is derived from \( |v_n| = \text{const} \) by the conformal mapping \( v_0 = \sigma(y_0), \sigma(0) = 0 \), and on this curve the consequences lie everywhere dense like as the consequences of \( v_n \) by \( v_n = v_0 e^{\alpha_n} \) on \( |v_n| = \text{const} \).

For a specially chosen system of iteration we have \( y_1 = e^{\alpha_n} y_0 + \ldots, y_n = e^{\alpha_n} y_0 + \ldots \) and the function \( v_n = v_0 e^{\alpha_n} \).
At least for such a system the consequents lie everywhere dense on a closed curve lying about the origin.

For each transformation \( v_n = v_n^{e^{\theta n}} \) we have, by the transformation \( y_n = s_n(v_n) \), and then by \( y_n^r = x_0 \) and \( y_n^r = z_n \) a sequence of transformed consequents by the iteration \( \varphi(x) \).

Since \( m \) is fixed, we have an everywhere dense set of consequents lying on some curves about the origin by the iteration of the function \( \varphi(x) \).

Thus in the case where \( \theta \) is an irrational number, either the consequents of a point converges to the origin, or there exists an infinite number of consequences of a point having a closed curve as a limiting set.

\[(\text{III d})\] 
\[|\alpha_n| = 1 \text{ and } \alpha_1 = e^{i\pi} \text{ where } \theta \text{ is a rational number, } \theta = \frac{q}{p}.\]

Then we must distinguish two cases.

(A) For some iterated functions, which are of the following form

\[y_1 = x + \ldots.
\]

or

\[y_{-1} = x + \ldots.
\]

we have \( y_1 \neq x \) or \( y_{-1} \neq x \).

(B) For all iterated functions, which are of the following form

\[y_1 = x + \ldots.
\]

and

\[y_{-1} = x + \ldots.
\]

we have \( y_1 = x \) and \( y_{-1} = x \).

Since \( y_0 = e^{i\pi} x + \ldots \) we must have \( y_1 = x + \ldots \) at least for the \( p \)-th iterated function of \( y \).

If \( y_1 = x + na' + \ldots \) and \( y_{-1} \neq x \), then we can apply the argument as in the case (III c) and we must have the antecedents of a branch point of \( y \) converging to the origin.

For, otherwise, the functions \( y_n = x + na' + \ldots \) being all monovalued and having the definite derivative at the origin, are limited in absolute values, and hence they form a normal family in some domain within \( V \).

Hence we can choose a sequence of functions which converges uniformly to a regula function. This is evidently impossible.

After Julia and Fatou it is shown that the antecedents and consequents of a point within \( V \) accumulate to the origin.

For some iterated functions combined from \( y_n, y_{-n} \ldots \) in another manner, we have
Here we may or may not have a function $y_0 = x + \ldots$.

In the first case, the above argument holds good. In the latter case, there occur an infinite number of times the functions of the form $y_i = e^{iax}x + \ldots$ where $a$ is the same for these functions.

Now consider the case (B).

If $y_i = x$, then there exists only a finite number of different functions about the origin. Let $y_1, y_2, \ldots, y_i$ be such functions, then $y_1, y_2, \ldots, y_i$ form a cyclic group.

For such functions, the $x$-plane about the origin can be divided into a finite number of parts each of which is transformed into another by some $y = e^{iax}x + \ldots$.

Thus in order that there may exist only a finite number of consequents and antecedents of a point in a vicinity of a point on the $x$-plane, it is necessary that the functions in the iteration either have no fixed point other than the infinity, or have only the fixed points of the property (III d) (B).

Now let

$$y_{ji} = e^{i(k+\lambda)x}y_k(1 + h_j(y_k))$$

be the functions which have the origin as a fixed point and have been deduced from the functions $x_1 = e^{i\lambda x}x_0 + \ldots$ similar to (3), and satisfying the condition (III d) (B).

Consider the iteration $y_{ij} = y_{ik}(\ldots y_{is}(y_k))$ in any manner, then either there may appear an infinite number of different iterated functions about the origin

$$y_i = e^{iax}y_0 + \ldots,$$

$i = 1, 2, \ldots$

or there may appear only a finite number of different functions. In the latter case we have

$$y_i = e^{iax}y_0 + \ldots = y_0.$$

Hence there exists only a finite number of different functions which have the origin as the fixed point.

In the former case let

$$w = t_i(y_0) = \frac{y_i}{e^{iax}} = y_0 + \ldots$$

be such an infinite number of functions, where $t_i$ are finite in number.
By the assumption $u_1$ are all mono-valued and hence they are limited in their absolute values, such as $A_1 < |u_1| < A_2$, where $A_1$ and $A_2$ are constants independent of the functions $u_1$.

Since $u_1$ are all different, the sequence of values, satisfying $A_1 < |t_1(y_1^{(n)})| < A_2$, are different, if we choose the sequence of values $y_1^{(1)}, y_1^{(2)}, \ldots, y_1^{(n)}, \ldots$ lying within the vicinity of the origin and having its limiting point at a point different from the origin.

For, if we cannot choose such a sequence $y_1^{(1)}, y_1^{(2)}, \ldots, y_1^{(n)}, \ldots$, then $t_1(y_1^{(n)}) = t_1(y_1^{(n)})$ for any such a sequence of $y_1$, which is impossible unless $t_1(y_1) = t_1(y_1)$.

Hence there must exist a limiting point of consequences of a point $y_1$ within $A_1 < |t_1| < A_2$.

Thus in order that there may exist only a finite number of consequences and antecedents of a point in a vicinity of a fixed point $O$ of the iteration of an element of an algebraic function $y=ax^3+\ldots$ when iterated by this element of the function in that vicinity, it is necessary and sufficient that there appears only a finite number of different functions in the vicinity of the fixed point $O$, when iterated in any manner by the sequence of the functions deduced from the function in the vicinity of the point, and the point $O$ must be the fixed point of the property (III d) (B) under the restriction (II).

II.

Rausenberger has shown that if there appears the function $y=x$ among the iterated functions of an algebraic function $y=\Phi(x)$, then the algebraic function must satisfy

$$y = \psi^{-1}e^{i\theta}\psi(x),$$

where $\psi$ is an algebraic function and $\theta$ is a rational number and further $y = \psi^{-1}e^{i\theta}\psi(x)$ is the function defined by eliminating $X$ between $P(X, y) = 0$ and $P \left( \frac{X}{e^{i\theta}}, x \right) = 0$, $P$ being a polynomial with respect to the two variables.

Hence a necessary condition, for which there may exist only a finite number of consequences and antecedents of a point in a vicinity of a fixed point $x=0$ in the theorem above mentioned, is that the algebraic function $\Phi(x)$ must be of the form

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Thus in order that the consequents and antecedents of the iteration of an algebraic function $\Phi(x)$ may form a properly discontinuous group in the whole $x$-plane it is necessary that either all the iterated functions $\Phi(x)$, $\Phi_{2}(x)$, $\ldots$, $\Phi_{n}(x)$, $\ldots$, $\Phi_{-1}(x)$, $\Phi_{-2}(x)$ $\ldots$ $\Phi_{-n}(x)$, $\ldots$ have no fixed point (except $\infty$) or $\Phi(x)$ may be of the form $\psi^{-1}e^{\alpha x}\psi(x)$ under the restriction (H).

We shall illustrate several cases by the following examples.

The algebraic function $P(y)=P(x)+c$, $P(x)$ being a polynomial of $x$, and $c$ a constant, has no fixed point (except $\infty$) for any iterated functions.

An irreducible factor of $P(y)=P(x)$ defines an algebraic function $y=\Phi(x)$ which generates only a finite number of iterated functions and hence it forms a finite discontinuous group.

The function $y^{2}=x+1$ shows us that though there exists only a finite number of consequents and antecedents about a fixed point $x_{0}=\frac{1}{(1-\alpha)^{\alpha}}$ in the iteration of the element $y-x_{0}=e^{i\frac{2\pi}{\alpha}}(x-x_{0})+\ldots$ there appears an infinite number of consequents and antecedents in the whole $x$-plane.

$y^{2}=x+1$ (or $y^{2}-\frac{1}{2}=-(-\frac{1}{2}x-\frac{1}{2})$, written in the form $y=\psi^{-1}(-\psi(x))$ forms a properly (infinite) discontinuous group, while $y^{2}=x+1$ does not form a discontinuous group.

Here we shall give some remark about the iteration of the second kind $\Phi(y)=\Phi(x)$.

The iteration must have a factor $y=x$, since it is defined by eliminating $z$ between $P(x,z)=0$, $P(y,z)=0$.

Fatou has shown that there exists only a finite number of fixed points at which $|R'(a)|\leq 1$ for the iteration of a rational function $y=R(x)$, $R(x)$ being of degree higher than 1 with respect to $x$.

He has considered as the iteration only the ordinary one.

If we consider the iteration of the second kind, then there may appear an infinite number of fixed points of the above property.

For example, we have for the iteration of $y=x^{2}$, $y_{n}=x^{2}=R_{n}(x)$. Hence $y^{2}=x^{2}$ has $2^{n}$ fixed points of such a property.

On the other hand there may appear an infinite number of fixed points for which $|\Phi'(a)|=1$ for the iteration of some algebraic function, even for the ordinary iteration.
For example, the iteration of \( y = (1 + \sqrt{y})^2 \), or \( y^2 + 2xy + x^2 + 1 = 0 \), has an infinite number of fixed points \( \frac{m^2}{4} \), \( m = 1, 2, \ldots \) for which \( \Phi'(\frac{m^2}{4}) = -1 \).

The consequents and antecedents of a point \( x_0 \) by the iteration of the second kind \( \Phi_n(y) = \Phi_n(x) \), \( n = 0, \pm 1, \pm 2, \ldots \) are a part of those of a point \( x_0 \) by the ordinary iteration operated in all possible manner.

But the functions appearing in the former iteration are not necessarily implied in those appearing in the latter iteration. This is evident by the example above mentioned.

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