Note on the Mean Convergence of a Sequence of Functions.

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I. The concept of the mean convergence was introduced by Professors M. Riesz and Fisher (1). Following them, the sequence of functions \( \{ f_n(x) \} \) whose \( p \)-th powers are integrable in Lebesgue's sense, is said to be mean convergent, when

\[
\lim_{n \to \infty} \int_0^1 |f_n(x) - f(x)|^p \, dx = 0, \quad p > 1;
\]

and the following theorem holds good:

The mean convergence of the sequence \( \{ f_n(x) \} \) is equivalent to the fact that there exists a function \( f(x) \) such that

\[
\lim_{n \to \infty} \int_0^1 |f(x) - f_n(x)|^p \, dx = 0.
\]

We will here consider a generalization of the above definition of the mean convergence.

Let \( g(x) \) be non-negative and integrable, and \( g \{ f_n(x) - f(x) \} \) also be integrable in Lebesgue's sense. If

\[
\lim_{n \to \infty} \int_0^1 g |f_n(x) - f(x)| \, dx = 0,
\]

then we say that the sequence \( \{ f_n(x) \} \) is mean convergent with respect to \( g(x) \). This was first considered by Mr. Noaillon (2) and further studied by Messrs. Kaczmarz and Nikliborc (3), while Mr. Isumi (4) gave more general results than those of Kaczmarz and Nikliborc.

II. Let \( g(x) \) be defined in \((a, b)\) and satisfy the conditions that

(i) \( g(x) \) is continuous and non-negative,
(ii) \( g(x) = g(-x) \),
(iii) there exist two constants \( M, N \) such that \( g(x) \geq M, \quad x \geq N \)

hold simultaneously.

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(1) See Hobson: Theory of functions of a real variable, II (1928).
and

(iv) \( g(x) \) tends to zero when and only when \( z \) tends to zero.

Mr. Isumi's theorem is the following:

If \( g(x) \) be conditioned as above, and

\[
\lim_{n \to \infty} \int g(f_n(x) - f(x)) \, dx = 0
\]

holds, there exists a function \( f(x) \) such that

\[
\lim_{n \to \infty} \int g(f_n(x) - f(x)) \, dx = 0.
\]

In this note, we will, corresponding to the theorem in I, consider the converse of this theorem. Then we can enunciate that the expressions (1) and (2) are equivalent to each other.

III. Lemma 1. Suppose that \( g(x) \) satisfies the conditions in II, and the integral

\[
\int g(f(x) - f_n(x)) \, dx
\]

exists and tends to zero as \( n \) tends to infinity. Then

\[
\int g(f_n(x) - f(x)) \, dx
\]

also exists for large \( m, n \).

Proof. We can suppose that

\[
\int g(f(x) - f_n(x)) \, dx < \varepsilon \text{ for } n \geq n_0,
\]

where \( \varepsilon \) is any assigned small quantity.

Assume that \( g(f(x) - f_n(x)) \geq \delta \) holds for \( |f(x) - f_n(x)| \geq \delta \), and all \( n \geq n_0 \). Since

\[
f_n(x) - f(x) = f_n(x) - f(x) + f(x) - f(x),
\]

the point in which

\[
|f_n(x) - f(x)| \geq \varepsilon \text{ for all } n \geq n_0
\]

belongs either to the set, in which

\[
|f_n(x) - f(x)| \geq \varepsilon - \delta \text{ for all } n \geq n_0
\]

or to the set, in which

\[
|f_n(x) - f(x)| \geq \delta \text{ for all } n \geq n_0.
\]

Then we can find \( H_m \) such that \( g(f_n(x) - f(x)) \geq H_m \) for \( |f_n(x) - f(x)| \geq \varepsilon \) \( \geq \varepsilon - \delta \). And let \( g(f_n(x) - f(x)) \geq H_m \) for \( |f_n(x) - f(x)| \geq H_m \) (for all \( n \geq n_0 \)). From the continuity of \( g(x) \), \( H_m \to \delta \) as \( \delta \to 0 \).
Define the function $g_m(x)$ such as
$$g_m(x) = g(x) \text{ when } g(x) < N,$$
$$= N \text{ when } g(x) \geq N,$$
and put $N = H_m' + 1$. Then still $g_m[f_n(x) - f_m(x)] \equiv H_m'$ for $f_n(x) - f_m(x) \equiv G_m$, and the set, in which $g_m[f_n(x) - f_m(x)] \equiv H_m'$ coincides with the set, in which $g[f_n(x) - f_m(x)] \equiv H_m'$. Therefore we have
$$m(E[g_m(x) - f_m(x)] \equiv H_m')) \leq m(E(g[f_n(x) - f_m(x)] \equiv H_m')) + m(E(g[f_n(x) - f_m(x)] \equiv z)).$$
By the hypothesis, $g|f(x) - f_m(x)|$ is integrable, and therefore putting
$$\int_a^b g[f(x) - f_m(x)] dx = L_m, \quad (L_m \text{ finite})$$
we have
$$m(E[g(x) - f_m(x)] \equiv H_m')) \leq \frac{L_m}{H_m}.$$
From (3) we get
$$m(E[g(x) - f_m(x)] \equiv z)) < \varepsilon.$$
Hence we get
$$m(E[g(x) - f_m(x)] \equiv H_m')) \leq \frac{L_m}{H_m} + \varepsilon.$$
Now suppose that
$$\int_a^b g[f_n(x) - f_m(x)] dx = \infty,$$
then from the definition of the integral,
$$\lim_{x \to +} \int_a^b g[f_n(x) - f_m(x)] dx = \infty.$$
And we can find $M_m' = M_m(N)$ such that $M_m' \varepsilon_x \to \infty$ as $N \to \infty$, where
$$\varepsilon_x = m(E[g(x) - f_m(x)] \equiv M_m')),$$
that is
$$M_m' \varepsilon_x > t/\varepsilon \text{ for } m \geq m_0.$$
If we put $H_m' = M_m'$, then $M_m' \to H_m'$ for $\delta \to 0$.
Therefore from (4), (6) and (7), we get
$$\frac{t}{M_m'} < m(E[g(x) - f_m(x)] > M_m')) < \frac{L_m}{H_m} + \varepsilon.$$
But we can find $m_0, n_0$ such that, for $m \geq m_0, n \geq n_0$, (8) is a contradiction.
III. Lemma 2. Suppose that \( g(x) \) be conditioned as in II, and for a given sequence \( \{f_n(x)\}, g[f(x)-f_n(x)] \) be integrable. Then provided

\[
\lim_{n \to \infty} \int_a^b g[f(x)-f_n(x)] \, dx = 0,
\]

we can find the subsequence of suffixes \( \{n_p\} \) such that \( f_n(x) \) converges almost always in \((a, b)\).

**Proof.** Let \( \varepsilon \) be a small positive number and \( \sum \delta_n \) be a convergent series with the sum \( \delta \). Then from the assumption, we can find \( \{n_p\} \) such that

\[
\int_a^b g[f(x)-f_{n_p}(x)] \, dx < (\delta \varepsilon)^2.
\]

The measure of the set \( E(g[f(x)-f_{n_p}(x)] \geq \delta \varepsilon) = E_\delta \) is less than \( \delta \varepsilon \). Let the compound set of \( E_1, E_2, \ldots, E_p, \ldots \) be \( E \), then \( m(E) < \sum \delta \varepsilon = \delta \). In the complementary set \( CE \) of \( E \) in \((a, b)\), we have

\[
g[f(x)-f_{n_p}(x)] < \delta \varepsilon.
\]

Therefore in \( CE \), \( g[f(x)-f_{n_p}(x)] \to 0 \), so \( f_{n_p}(x) \to f(x) \) as \( p \to \infty \) from the continuity of \( g(x) \).

Since \( \varepsilon \) is arbitrary, \( f_{n_p}(x) \) is almost always convergent to \( f(x) \) in \((a, b)\).

**IV.** We will now prove the

**Theorem.** If \( g(x) \) is conditioned as in II, and

\[
\lim_{n \to \infty} \int_a^b g[f(x)-f_n(x)] \, dx = 0,
\]

then

\[
\lim_{n \to \infty} \int_a^b g[f(x)-f_n(x)] \, dx = 0.
\]

Assume that the theorem does not hold, then there exist sequences \( (m_1), (m_2) \) and a positive number \( \xi \) such that

\[
\lim_{k \to \infty} \int_a^b g[f_{m_k}(x)-f_{m_2}(x)] \, dx = \xi > 0.
\]

From the assumption that

\[
\lim_{k \to \infty} \int_a^b g[f_{m_k}(x)-f(x)] \, dx = 0,
\]

we can extract the subsequence \( \{f_{m_k}(x)\} \) from \( \{f_{m_k}(x)\} \) such that

\[
\lim_{k \to \infty} f_{m_k}(x) = f(x)
\]
is valid almost always.

Now again we can find the subsequence \( |f_{n_k}(x)| \) in \( |f_n(x)| \) such that

\[
\lim_{k \to \infty} f_{n_k}(x) = f(x)
\]

holds almost always.

Therefore \( |f_{n_k}(x)|, |f_{n_k}(x)| \) are, at the same time, convergent to \( f(x) \) almost always as \( k'' \to \infty \).

Now from the well known Egoroff's theorem

\[
f_{n_k}(x) - f(x) < \frac{\varepsilon}{2} \quad \text{for } x \in K,
\]

\[
f_{n_k}(x) - f(x) < \frac{\varepsilon}{2} \quad \text{for } x \in K
\]

for \( k'' > k_n \), where \( K \) is a subset of \((a, b)\) and the measure of \( CK \) is less than any assigned quantity \( \eta \).

Therefore in \( K \), we have

\[
f_{n_k}(x) - f_{n_k}(x) < \varepsilon \quad \text{for } k'' > k_n,
\]

consequently

\[
g|f_{n_k}(x) - f_{n_k}(x)| < \delta \quad \text{for } k'' > k_n.
\]

And

\[
\int_a^b g|f_{n_k}(x) - f_{n_k}(x)| \, dx = \int_K g|f_{n_k}(x) - f_{n_k}(x)| \, dx + \int_{\mathbb{X}} g|f_{n_k}(x) - f_{n_k}(x)| \, dx
\]

\[
\leq m(K')\delta + \eta \int_a^b g|f_{n_k}(x) - f_{n_k}(x)| \, dx
\]

\[
\leq m(K')\delta + M\eta \quad \text{for } k'' > k_n,
\]

where \( M = \max_{k'' > k_n} \int_a^b g|f_{n_k}(x) - f_{n_k}(x)| \, dx \).

Let \( \delta, \eta \) be so small that \( m(K')\delta + M\eta < \xi \), then this contradicts (9). Thus the theorem is proved.

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